Electronic Transactions on Numerical Analysis.
Volume 46, pp. 1–xx, 2017.

ISSN 1068–9613.

ETNA
Kent State University and
Johann Radon Institute (RICAM)

ANALYSIS OF THE TRUNCATION ERROR AND BARRIER-FUNCTION TECHNIQUE FOR A BAKHVALOV-TYPE MESH*

THÁI ANH NHAN† AND RELJA VULANOVIĆ‡

Abstract. We use a barrier-function technique to prove the parameter-uniform convergence for singularly perturbed convection-diffusion problems discretized on a Bakhvalov-type mesh. This is the first proof of this kind in the research literature, the barrier-function approach having only been applied so far to Shishkin-type meshes.

Key words. singular perturbation, convection-diffusion, boundary-value problem, Bakhvalov-type meshes, finite differences, uniform convergence

AMS subject classifications. 65L10, 65L12, 65L20, 65L70.

1. Introduction.

Singularly perturbed boundary-value problems arise as models of various phenomena in science and engineering [4, 5, 18]. Their numerical solution represents a challenge because of the presence of boundary and/or interior layers in the continuous solution. This is why layer-adapted meshes are often used in numerical methods for solving these problems. Bakhvalov and Shishkin meshes and their modifications are the most well-known meshes of this kind. Despite the fact that the latter [20] was introduced about two decades later than the former [3], Shishkin-type (S-type) meshes have become predominant in the singular-perturbation research, largely due to their simple construction. We refer the reader to the monographs [11, 18] and many references therein for dedicated discussions on such layer-adapted meshes and relevant numerical schemes.

The original Shishkin mesh and its slight modifications enjoy properties of piecewise-uniformity and explicitly defined transition points between fine and coarse parts of the mesh, which greatly simplifies the analysis of numerical methods applied. Their drawback is that they produce errors which contain $\ln N$-factors, where N is the number of mesh steps. Bakhvalov-type (B-type) meshes do not suffer from this as they are graded in the layer and smoothly transition to a coarser part at a point which is farther away from the layer than the Shishkin transition point. B-type meshes include the original Bakhvalov mesh and its modifications, like the one due to Kopteva [6, 7] (or the variation in [18, p. 120]), as well as Vulanović's generalization [22]. The generalization in [11, 17] combines the Shishkin transition point with the possibility of having a smoothly graded mesh in the layer, which also eliminates $\ln N$-factors from the error. Because of the transition point, these meshes are considered S-type meshes. Their graded part can be generated by the same mesh-generating functions like those used to create the graded part of B-type meshes. In this sense, we have meshes like the Bakhvalov-Shishkin mesh or the Vulanović-Shishkin mesh.

Generally speaking, the nature of singular perturbation problems makes their numerical analysis difficult. For instance, when convection-dominated problems are solved by a finite-difference method on a layer-adapted mesh, special techniques are needed to prove that the method converges uniformly with respect to the singular perturbation parameter ε. Those techniques include, but are not limited to, the use of barrier-function estimates of the truncation error (e.g., [16, 17, 21]); of the hybrid stability-inequalities [1, 2, 6, 7, 12], which usually

*Received: Oct. 7, 2018 Accepted... Published online on... Recommended by....
†Department of Mathematics and Science, Holy Names University, 3500 Mountain Blvd., Oakland, CA 94619, USA; Corresponding author; E-mail: nhan@hnu.edu.
‡Department of Mathematical Sciences, Kent State University at Stark, 6000 Frank Ave. NW, North Canton, OH 44720, USA; E-mail: rvulanov@kent.edu

involves the discrete Green's function; or of the newly proposed preconditioning approach [13, 14, 15, 26].

All the above ε-uniform convergence proof techniques on layer-adapted meshes have been applied successfully to singularly perturbed convection-diffusion problems, except for one case, the truncation error and barrier function technique on B-type meshes. The question whether this kind of proof would be possible was posed in the early 2000s, and has remained unanswered until now (see the survey [8, p. 1068], for instance). At the same time, the existing barrier-function technique works for S-type meshes, even the ones graded in the layer [17]. This is mainly because the step sizes of S-type meshes possess certain technical properties, which B-type meshes do not have. For example, all mesh steps in the layer region of the S-type meshes introduced in [17] are bounded by $C\varepsilon$, where C denotes a positive generic constant independent of ε and N. By contrast, as will be seen in the present paper, B-type meshes do not satisfy this property. This is why other techniques (i.e., the hybrid stability-inequalities and preconditioning) have been developed for B-type meshes.

In this paper, we close the existing theoretical gap and provide the first-ever barrier-function proof of ε-uniform convergence for convection-diffusion problems discretized on a B-type mesh. The construction of an appropriate barrier function is based on a careful and detailed analysis of the truncation error. Our proof technique is inspired by the one we recently used in [16], where we modified and improved the barrier-function approach to prove ε-uniform convergence for a generalized Shishkin mesh.

To present our proof, we consider the simplest model problem, the simplest finite-difference scheme, and one of the simplest B-type meshes. For the model problem, we take a linear singularly perturbed convection-diffusion problem in one dimension,

$$\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x),\ x \in (0,1),\ u(0) = u(1) = 0, \tag{1.1}$$

with a small positive perturbation parameter ε and $C^1[0,1]$-functions b, c, and f, where b and c satisfy

$$b(x) \geq \beta > 0,\ \ c(x) \geq 0\ \text{ for } x \in I := [0,1].$$

With these assumptions, the problem (1.1) has a unique solution, u, which in general has an exponential boundary layer near $x = 0$. We discretize the problem on the simplest of Vulanović's modifications of the original Bakhvalov mesh, [22], and use the standard upwind scheme, the simplest ε-uniformly stable scheme available. This method provides ε-uniform convergence of the first order, which is what we prove. We particularly emphasize that this result is not new; the same has been proved using the other above-mentioned techniques [1, 2, 15]. Rather, the novel contribution of the paper lies in the way the problem is analyzed by a barrier-function approach for the first time. Our analysis provides more understanding of the behavior of the truncation error at every discretization point of the B-type mesh. And, therefore, we provide the theoretical answer to the open problem posed in [8, p. 1068] and also [11, Remark 4.21].

Additionally, we are motivated to study the barrier-function technique because it can be generalized to two dimensions [9, 10]. This has been done on S-type meshes, but the question whether one can prove ε-uniform convergence on a B-type mesh for the upwind discretization of the two dimensional convection-diffusion problem, [19, Question 4.1], has been open until now. We demonstrate in the appendix that our approach can be extended to two dimensions, thus answering the question affirmatively.

The rest of the paper is organized as follows. The B-type mesh is introduced and analyzed in Section 2. This is followed in Section 3 by the truncation-error analysis. We then prove the main result, ε-uniform convergence, in Section 4, and illustrate it with a numerical experiment

ETNA
Kent State University and
Johann Radon Institute (RICAM)

in Section 5. Some concluding discussions are provided in Section 6, which continues in the appendix, where we comment on the two-dimensional case of the problem.

2. The Vulanović-Bakhvalov mesh. Let x_i, $i = 0, 1, \ldots, N$, denote the points of the discretization mesh, $0 = x_0 < x_1 < \cdots < x_N = 1$, and let $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. We also define $\hbar_i = (h_i + h_{i+1})/2$, $i = 1, 2, \ldots, N-1$. A mesh function on the discretization mesh is denoted by $w^N = (w_0^N, w_1^N, \ldots, w_N^N)$. If g is a function defined on I, we write g_i instead of $g(x_i)$ and g^N for the corresponding mesh function.

We repeat that C denotes a generic positive constant independent of both ε and N. Some specific constants of this kind will be indexed.

The design and generalization of the Shishkin meshes have gained much attention from researchers, see [16, 17, 23, 24] for instance. In contrast, there has been less attention on the generalized construction or analysis of the Bakhvalov mesh. A rare example of the generalization of the Bakhvalov mesh is an early contribution by Vulanović, [22]. A modification of the Bakhvalov mesh can also be found in [6, 7]. The mesh points x_i of any B-type mesh are generated by a function λ in the sense that $x_i = \lambda(t_i)$, where $t_i = i/N$ for $i = 0, 1, \ldots, N$. The mesh-generating function λ is defined as follows:

$$\lambda(t) = \begin{cases} \psi(t), & t \in [0, \alpha], \\ \psi(\alpha) + \psi'(\alpha)(t - \alpha), & t \in [\alpha, 1], \end{cases} \tag{2.1}$$

with $\psi = a\varepsilon\phi$, where a is a positive fixed mesh-parameter and ϕ is a smooth function which, essentially, is the inverse of the exponential-layer function. The point α is the solution of the equation

$$\psi(\alpha) + \psi'(\alpha)(1 - \alpha) = 1. \tag{2.2}$$

The part of λ on the interval $[0, \alpha]$ generates the fine portion of the mesh in the layer, while the part on $[\alpha, 1]$ generates the coarse mesh outside the layer (this part is the tangent line from the point $(1, 1)$ to ψ, touching ψ at $(\alpha, \psi(\alpha))$).

In [22], the author also introduces a simpler B-type mesh, in which a Padé-approximation of the exponential-layer function is employed to construct the function ϕ,

$$\phi(t) = \frac{t}{q - t} = \frac{q}{q - t} - 1, \quad t \in [0, \alpha],$$

where q is another fixed mesh-parameter, $0 < q < 1$. We consider this mesh in the present paper and call it the Vulanović-Bakhvalov mesh (VB mesh). With this choice of ϕ, the equation (2.2) reduces to a quadratic one and its solution is easy to find,

$$\alpha = \frac{q - \sqrt{a\varepsilon q(1 - q + a\varepsilon)}}{1 + a\varepsilon}.$$

The two mesh-parameters have to satisfy $a\varepsilon < q$ (which is equivalent to $\psi'(0) < 1$) and then α is positive. Note also that $\alpha < q$ and

$$q - \alpha = \zeta\sqrt{\varepsilon}, \quad \zeta \leq C, \quad \frac{1}{\zeta} \leq C.$$

Let J be the index such that $t_{J-1} < \alpha \leq t_J$. Starting from the mesh point x_J, the mesh is uniform. However, x_J behaves differently from the transition point of the Shishkin mesh because

$$x_J \geq \psi(\alpha) = \frac{a\alpha}{\zeta}\sqrt{\varepsilon}. \tag{2.3}$$

Additionally, it is also worth mentioning that λ defined in (2.1) is differentiable on $[0, 1]$, whereas mesh-generating functions of graded S-type meshes are only piecewise differentiable.

We now derive some estimates for the VB mesh steps.

LEMMA 2.1. *Let $t_J \le q$. Then we have the following estimates for the step sizes of the VB mesh in the layer region:*

$$C\varepsilon N^{-1} \le h_i \le C\varepsilon N, \quad i = 1, 2, \dots, J-1, \tag{2.4}$$

and

$$1 \le \frac{h_i}{h_{i-1}} \le 3, \quad i = 2, 3, \dots, J-1. \tag{2.5}$$

Furthermore, when $h_i > \varepsilon$, we have that

$$\frac{1}{q - t_{i-1}} > \frac{N^{1/2}}{\sqrt{2aq}}, \quad i = 1, 2, \dots, J-1. \tag{2.6}$$

Proof. First, we show that

$$C\varepsilon N^{-1} \le h_i, \quad i = 1, 2, \dots, J-1.$$

This is because

$$\begin{aligned} h_i = x_i - x_{i-1} &= a\varepsilon[\phi(t_i) - \phi(t_{i-1})] \ge a\varepsilon N^{-1}\phi'(t_{i-1}) \\ &= a\varepsilon N^{-1}\frac{q}{(q - t_{i-1})^2} \ge a\varepsilon N^{-1}\frac{q}{(q)^2} \ge C\varepsilon N^{-1}. \end{aligned}$$

On the other hand, for $i \le J-1$ and $t_J \le q$, we have

$$\begin{aligned} h_i = x_i - x_{i-1} = a\varepsilon[\phi(t_i) - \phi(t_{i-1})] &= \frac{a\varepsilon q}{N(q - t_{i-1})(q - t_i)} \\ \le \frac{a\varepsilon q}{N(q - t_{J-2})(q - t_{J-1})} &\le \frac{a\varepsilon q}{N(t_J - t_{J-2})(t_J - t_{J-1})} \le C\varepsilon N. \end{aligned}$$

To prove (2.5), the mesh construction yields $h_{i-1} \le h_i$, so that we immediately have $\frac{h_i}{h_{i-1}} \ge 1$. Furthermore,

$$\frac{h_i}{h_{i-1}} = \frac{q - t_{i-2}}{q - t_i} = \frac{q - t_i + 2/N}{q - t_i} \le 1 + \frac{2}{N(t_J - t_{J-1})} = 3$$

for all $i \le J-1$.

To verify (2.6), we first observe that for $i \le J-1$ and $t_J \le q$:

$$\frac{q - t_{i-1}}{q - t_i} = 1 + \frac{1}{N(q - t_i)} \le 1 + \frac{1}{N(t_J - t_i)} \le 2.$$

That is,

$$\frac{1}{q - t_i} \le \frac{2}{q - t_{i-1}}. \tag{2.7}$$

Additionally, when $h_i > \varepsilon$, we have

$$\frac{h_i}{\varepsilon} = \frac{aq}{N(q - t_{i-1})(q - t_i)} > 1$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

and then (2.7) yields

$$\frac{2aq}{N(q-t_{i-1})^2} > 1,$$

which asserts (2.6). $\square$

REMARK 2.2. The mesh step-size estimates stated in (2.4) distinguish our B-type mesh from S-type meshes in the sense of [14, 17]. That is, in the layer region, while the step size h_i of S-type meshes is bounded from above by $C\varepsilon$, that of the VB mesh is gradually graded with $h_1 \sim \mathcal{O}(\varepsilon N^{-1})$ and $h_{J-1} \sim \mathcal{O}(\varepsilon N)$. This is a striking contrast between the two mesh types.

REMARK 2.3. Without the condition $t_J \le q$, all the estimates in Lemma 2.1 are true for $i \le J-2$.

We now consider step-size estimates for the case $q < t_J$. We also define $t_{J-1/2} = \dfrac{t_{J-1}+t_J}{2} = \dfrac{J-1/2}{N}$.

LEMMA 2.4. *Let $q < t_J$. Then the following estimates are satisfied:*

- *When $\alpha \le t_{J-1/2}$, we have*

$$h_J \ge CN^{-1}. \tag{2.8}$$

- *When $t_{J-1/2} < \alpha$, we have*

$$h_{J-1} \le C\varepsilon N. \tag{2.9}$$

Proof. First, consider $\alpha \le t_{J-1/2}$. Then, $h_J = \gamma_1 + \gamma_2$, where $\gamma_1 = x_\alpha - x_{J-1}$ and $\gamma_2 = x_J - x_\alpha$ with $x_\alpha = \psi(\alpha)$. It follows that

$$\begin{aligned} h_J \ge \gamma_2 = \psi'(\alpha)(t_J - \alpha) &= \frac{a\varepsilon q}{q-\alpha}\left(\frac{t_J-\alpha}{q-\alpha}\right) \\ &\ge \frac{aq}{\zeta^2}\left(t_J - t_{J-1/2}\right) \ge \frac{aq}{\zeta^2}\frac{1}{2N}, \end{aligned}$$

which gives (2.8).

Second, for $t_{J-1/2} < \alpha$ and because $t_{J-1} < t_{J-1/2} < \alpha < q$, we have

$$\begin{aligned} h_{J-1} &= \frac{a\varepsilon q}{N(q-t_{J-2})(q-t_{J-1})} \\ &\le \frac{a\varepsilon q}{N(t_{J-1}-t_{J-2})(t_{J-1/2}-t_{J-1})} \le 2a\varepsilon q N. \end{aligned}$$

$\square$

3. The upwind discretization and truncation-error estimate. We discretize the problem (1.1) on the VB mesh using the upwind finite-difference scheme,

$$w_0^N = 0,$$

$$\mathcal{L}^N w_i^N := -\varepsilon D'' w_i^N - b_i D^+ w_i^N + c_i w_i^N = f_i, \quad i = 1, 2, \ldots, N-1, \tag{3.1}$$

$$w_N^N = 0,$$

where

$$D''w_i^N = \frac{1}{\hbar_i}\left(D^+w_i^N - D^-w_i^N\right)$$

and

$$D^+w_i^N = \frac{w_{i+1}^N - w_i^N}{h_{i+1}}, \quad D^-w_i^N = \frac{w_i^N - w_{i-1}^N}{h_i}.$$

It is easy to see that the operator $\mathcal{L}^N$ satisfies the discrete maximum principle. Therefore, the discrete problem (3.1) has a unique solution w^N.

We proceed to provide the truncation-error estimate when the problem (1.1) is discretized by the above upwind scheme on the VB mesh. Let

$$\tau_i[g] = \mathcal{L}^N g_i - (\mathcal{L}g)_i, \;\; i = 1, 2, \dots, N-1,$$

for any $C^2(I)$-function g. In particular, $\tau_i[u]$ is the truncation error of the finite-difference operator $\mathcal{L}^N$ and

$$\tau_i[u] = \mathcal{L}^N u_i - \mathcal{L}^N w_i^N = \mathcal{L}^N(u - w^N)_i. \tag{3.2}$$

By Taylor's expansion we get that

$$|\tau_i[u]| \le Ch_{i+1}(\varepsilon\|u'''\|_i + \|u''\|_i), \tag{3.3}$$

where $\|g\|_i := \max_{x_{i-1}\le x\le x_{i+1}} |g(x)|$ for any $g \in C(I)$.

We estimate the truncation error below by using the following decomposition of the continuous solution u into the smooth and boundary-layer parts, [11, Theorem 3.48]:

$$u(x) = s(x) + y(x),$$

where for $x \in I$ and $k = 0, 1, 2, 3$ we have

$$|s^{(k)}(x)| \le C\left(1 + \varepsilon^{2-k}\right) \tag{3.4}$$

and

$$|y^{(k)}(x)| \le C\varepsilon^{-k}e^{-\beta x/\varepsilon}. \tag{3.5}$$

In addition, the layer component, y, satisfies a homogeneous differential equation,

$$\mathcal{L}y(x) = 0, \quad x \in (0, 1). \tag{3.6}$$

Moreover, in the proof of the next lemma, we crucially need the inequality

$$e^{-\beta x_i/(2\varepsilon)} = \prod_{j=1}^{i}\left(e^{-\beta h_j/(2\varepsilon)}\right) \le \prod_{j=1}^{i}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} =: \bar{y}_i^N \tag{3.7}$$

for $i = 0, 1, \dots, N$, which is based on $e^t \ge 1 + t$, for all $t \ge 0$.

LEMMA 3.1. *The truncation error for the regular part satisfies*

$$|\tau_i[s]| \le CN^{-1}, \quad i = 1, 2, \dots, N-1.$$

The layer part can be estimated as follows:

ETNA
Kent State University and
Johann Radon Institute (RICAM)

- *For $i \geq J+1$, we have*

$$|\tau_i[y]| \leq CN^{-1}. \tag{3.8}$$

- *For $i \leq J$, we have the following subcases:*
  - *When $h_i \leq \varepsilon$, we have*

$$|\tau_i[y]| \leq \begin{cases} CN^{-1}, & i = J \\ C\varepsilon^{-1}\bar{y}_i^N N^{-1}, & i \leq J-1. \end{cases} \tag{3.9}$$

  - *When $h_i > \varepsilon$, we first have an estimate for $i = J$,*

$$|\tau_i[y]| \leq CN^{-1}. \tag{3.10}$$

For $i \leq J-1$ and $t_J \leq q$,

$$|\tau_i[y]| \leq Ch_{i+1}^{-1}\bar{y}_i^N N^{-1}. \tag{3.11}$$

For $i \leq J-1$ and $q < t_J$, we have

$$|\tau_i[y]| \leq \begin{cases} CN^{-1}, & i = J-1 \quad \& \quad \alpha \leq t_{J-1/2}, \\ Ch_{i+1}^{-1}\bar{y}_i^N N^{-1}, & i = J-1 \quad \& \quad t_{J-1/2} < \alpha, \\ Ch_{i+1}^{-1}\bar{y}_i^N N^{-1}, & i \leq J-2. \end{cases} \tag{3.12}$$

Proof. It is an easy computation to bound $|\tau_i[s]|$ applying (3.3) to s and using the estimates in (3.4). Similarly, for the layer component, we use the derivative estimates (3.5).

To prove (3.8), we apply (3.3) to y and note that in this case $t_{i-1} \geq t_J \geq \alpha$. Then we have

$$\begin{aligned} |\tau_i[y]| &\leq Ch_{i+1}\left(\varepsilon\|y'''\|_i + \|y''\|_i\right) \leq CN^{-1}\lambda'(t_{i+1})\varepsilon^{-2}e^{-\beta\lambda(t_{i-1})/\varepsilon} \\ &\leq CN^{-1}\lambda'(t_{i+1})\varepsilon^{-2}e^{-\beta\lambda(\alpha)/\varepsilon} \leq CN^{-1}\varepsilon^{-2}e^{-a\beta\alpha/(\zeta\sqrt{\varepsilon})} \leq CN^{-1}, \end{aligned}$$

where we have used (2.3) and the fact that $\varepsilon^{-2}e^{-a\beta\alpha/(\zeta\sqrt{\varepsilon})} \leq C$.

The case for which $i = J$ and $h_J \leq \varepsilon$, we have

$$\begin{aligned} |\tau_J[y]| &\leq Ch_{J+1}\left(\varepsilon\|y'''\|_J + \|y''\|_J\right) \leq CN^{-1}\lambda'(t_{J+1})\varepsilon^{-2}e^{-\beta x_{J-1}/\varepsilon}, (\lambda'(t_{J+1}) \leq \phi'(\alpha) \leq C) \\ &\leq CN^{-1}\varepsilon^{-2}e^{-\beta x_J/\varepsilon} \leq CN^{-1}\varepsilon^{-2}e^{-\beta\lambda(\alpha)/\varepsilon} \leq CN^{-1}\varepsilon^{-2}e^{-a\beta\alpha/(\zeta\sqrt{\varepsilon})} \leq CN^{-1}. \end{aligned}$$

The case for which $i \leq J-1$ and $h_i \leq \varepsilon$ is handled as follows. In order to show (3.9), we divide this case into two subcases:

1. When $t_{i-1} \leq q - 3/N$.
2. When $q - 3/N < t_{i-1} < \alpha$.

Subcase 1. Note that, when $t_{i-1} \leq q - 3/N$, we have

$$t_{i+1} \leq q - 1/N < q, \qquad (\text{so } \lambda'(t_{i+1}) \leq a\varepsilon\phi'(t_{i+1}))$$

and

$$q - t_{i+1} = q - t_{i-1} - \frac{2}{N} = \frac{1}{3}(q - t_{i-1}) + \frac{2}{3}(q - t_{i-1}) - \frac{2}{N} \geq \frac{1}{3}(q - t_{i-1}),$$

and also,

$$q - t_i = q - t_{i-1} - 1/N = \frac{2}{3}(q - t_{i-1}) + \frac{1}{3}(q - t_{i-1}) - \frac{1}{N} \geq \frac{2}{3}(q - t_{i-1}),$$

because $q - t_{i-1} \geq 3/N$ yields $\frac{1}{3}(q - t_{i-1}) - 1/N \geq 0$. Hence,

$$\begin{aligned}
|\tau_i[y]| &\leq Ch_{i+1}\left(\varepsilon\|y'''\|_i + \|y''\|_i\right) \\
&\leq CN^{-1}\lambda'(t_{i+1})\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon} \\
&\leq C\varepsilon^{-1}N^{-1}\phi'(t_{i+1})e^{-\beta x_{i-1}/2\varepsilon}e^{-\beta x_{i-1}/2\varepsilon} \\
&\leq C\varepsilon^{-1}N^{-1}(q - t_{i+1})^{-2}e^{-\frac{a\beta}{2}(q/(q-t_{i-1})-1)}e^{-\beta x_{i-1}/2\varepsilon} \\
&\leq C\varepsilon^{-1}N^{-1}(q - t_{i-1})^{-2}e^{-a\beta q/[2(q-t_{i-1})]}e^{-\beta x_{i-1}/2\varepsilon} \\
&\leq C\varepsilon^{-1}N^{-1}(q - t_{i-1})^{-2}e^{-a\beta q/[2(q-t_{i-1})]}e^{-\beta x_i/(2\varepsilon)}e^{\beta h_i/(2\varepsilon)} \\
&\leq C\varepsilon^{-1}N^{-1}(q - t_{i-1})^{-2}e^{-a\beta q/[2(q-t_{i-1})]}\bar{y}_i^N,
\end{aligned}$$

where we used $h_i \leq \varepsilon$ in the last step. It follows from there that

$$|\tau_i[y]| \leq C\varepsilon^{-1}N^{-1}\bar{y}_i^N$$

because $(q - t_{i-1})^{-2}e^{-a\beta q/(q-t_{i-1})} \leq C$.

Subcase 2. This can be handled as follows:

$$\begin{aligned}
|\tau_i[y]| &\leq 2\varepsilon\|y''\|_i + 2\|y'\|_i \leq C\varepsilon^{-1}e^{-\beta x_{i-1}/2\varepsilon}e^{-\beta x_{i-1}/2\varepsilon} \\
&\leq C\varepsilon^{-1}e^{-\beta x_i/(2\varepsilon)}e^{\beta h_i/(2\varepsilon)}e^{-\beta x_{i-1}/(2\varepsilon)} \\
&\leq C\varepsilon^{-1}\bar{y}_i^N e^{-\beta x_{i-1}/2\varepsilon} \leq C\varepsilon^{-1}\bar{y}_i^N e^{-\frac{a\beta}{2}(qN/3-1)} \leq C\varepsilon^{-1}\bar{y}_i^N N^{-1}.
\end{aligned}$$

This completes the proof of (3.9).

We next proceed to the case when $h_i > \varepsilon$. For the estimate (3.10), we consider the truncation error in the form $\tau_J[y] = \mathcal{L}^N y$, which is valid because of (3.6). Thus, we have

$$|\tau_J[y]| \leq P_J + Q_J + R_J,$$

where

$$P_J = \varepsilon|D''y_J|, \quad Q_J = b_J|D'y_J|, \text{ and } R_J = c_J|y_J|.$$

For P_J, since $\hbar_J \geq h_{J+1}/2 \geq CN^{-1}$, we get $\hbar_J^{-1} \leq CN$ and

$$\begin{aligned}
P_J &\leq C\hbar_J^{-1}e^{-\beta x_{J-1}/\varepsilon} \leq CNe^{-\beta a\phi(t_{J-2})} \\
&\leq CNe^{-\beta aq/(q-t_{J-1})}.
\end{aligned} \tag{3.13}$$

Note that when $\varepsilon < h_J$, it implies $\varepsilon < CN^{-1}$ or $\sqrt{\varepsilon} < CN^{-1/2}$. Therefore,

$$q - t_{J-1} = (q - \alpha) + (\alpha - t_{J-1}) \leq \zeta\sqrt{\varepsilon} + N^{-1} \leq CN^{-1/2}.$$

Plugging this inequality in the estimate for P_J gives

$$P_J \leq CNe^{-\beta aq/(q-t_{J-1})} \leq CNe^{-C\sqrt{N}} \leq CN^{-1}.$$

Similar arguments work for Q_J and R_J.

We now move onto the case $i \leq J - 1$ and $t_J \leq q$. We have

$$\begin{aligned}
R_i &\leq Ce^{-\beta x_i/\varepsilon} = Ch_{i+1}^{-1}h_{i+1}e^{-\beta x_i/(2\varepsilon)}e^{-\beta x_i/(2\varepsilon)} \\
&\leq Ch_{i+1}^{-1}\bar{y}_i^N e^{-\beta x_i/(2\varepsilon)} \leq Ch_{i+1}^{-1}\bar{y}_i^N e^{-\beta x_{i-1}/(2\varepsilon)} \\
&\leq Ch_{i+1}^{-1}\bar{y}_i^N e^{-\beta aq/[2(q-t_{i-1})]} \leq Ch_{i+1}^{-1}\bar{y}_i^N e^{-\beta\sqrt{aq}N^{1/2}/(2\sqrt{2})} \\
&\leq Ch_{i+1}^{-1}\bar{y}_i^N N^{-1}.
\end{aligned} \tag{3.14}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

For P_i, and analogously for Q_i,

$$\begin{aligned} P_i &\le C\hbar_i^{-1} e^{-\beta x_{i-1}/\varepsilon} \le C h_{i+1}^{-1} e^{-\beta x_{i-1}/(2\varepsilon)} e^{-\beta x_{i-1}/(2\varepsilon)} \\ &\le C h_{i+1}^{-1} \bar{y}_i^N \left(1 + \frac{\beta h_i}{2\varepsilon}\right) e^{-\beta x_{i-1}/(2\varepsilon)}. \end{aligned}$$

We then use the inequality $h_i \le C\varepsilon N$ from Lemma 2.1 and get

$$\begin{aligned} P_i &\le C h_{i+1}^{-1} \bar{y}_i^N (1 + CN) e^{-\beta a q/[2(q - t_{i-1})]} \\ &\le C h_{i+1}^{-1} \bar{y}_i^N (1 + CN) e^{-\beta\sqrt{aq} N^{1/2}/(2\sqrt{2})} \le C h_{i+1}^{-1} \bar{y}_i^N N^{-1}. \end{aligned} \tag{3.15}$$

This asserts (3.11).

For $h_i > \varepsilon$, $i = J-1$ and $\alpha \le t_{J-1/2}$, using (2.8) we get

$$\begin{aligned} P_{J-1} &\le C\hbar_{J-1}^{-1} e^{-\beta x_{J-2}/\varepsilon} \le C h_J^{-1} e^{-\beta x_{J-2}/\varepsilon} \\ &\le C N e^{-\beta\sqrt{aq} N^{1/2}/\sqrt{2}} \le C N^{-1}. \end{aligned}$$

We apply a similar argument to Q_{J-1} and R_{J-1}. This implies that $|\tau_{J-1}[y]| \le CN^{-1}$, which is the first case in (3.12).

For $h_i > \varepsilon$ with $i = J-1$ and $t_{J-1/2} < \alpha$, we use (2.9) to verify that the estimates in (3.14) and (3.15) are true. In this way we prove the second case in (3.12).

For the last case in (3.12), the assertions of Lemma 2.1 are satisfied for $i \le J-2$ (see also Remark 2.3), so the analysis of the truncation error in this case is the same as that of the estimate (3.11). ☐

The detailed estimates of the truncation errors of the regular and layer components in Lemma 3.1, invoking $\tau_i[u] = \tau_i[s] + \tau_i[y]$, can be summarized as follows.

THEOREM 3.2. *The truncation error of the upwind discretization of the problem* (1.1) *on the VB mesh satisfies the following:*

- *When $h_i > \varepsilon$,*

$$|\tau_i[u]| \le C\left(N^{-1} + h_{i+1}^{-1} \bar{y}_i^N N^{-1}\right), \quad i = 1, 2, \ldots, N-1,$$

- *When $h_i \le \varepsilon$,*

$$|\tau_i[u]| \le C\left(N^{-1} + \varepsilon^{-1} \bar{y}_i^N N^{-1}\right), \quad i = 1, 2, \ldots, N-1,$$

where $\bar{y}_i^N$ is defined in (3.7).

The above theorem is a crucial component of our analysis. It is interesting to mention that similar estimates can be found in the truncation-error bound (4.9) in [6]. However, this bound is obtained for the central scheme applied on a mesh different from ours (a slight modification of the Bakhvalov mesh) and is used in a discrete-Green's-function approach, not a barrier-function one.

4. The barrier function and ε-uniform convergence.

In this section, we propose a barrier function to bound the truncation error established in Theorem 3.2. We then apply the discrete maximum principle to get the ε-uniform-convergence result.

Imitating the newly proposed barrier function in [16], we form

$$\gamma_i = \gamma_i^{(1)} + \gamma_i^{(2)}, \quad i = 0, 1, \ldots, N,$$

with

$$\gamma_i^{(1)} = C_1 N^{-1}(1 - x_i) \quad \text{and} \quad \gamma_i^{(2)} = C_2 \bar{y}_i^N N^{-1},$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

where C_1 and C_2 are appropriately chosen positive constants independent of both ε and N.

LEMMA 4.1. *There exist sufficiently large constants C_1 and C_2 such that*

$$\mathcal{L}^N\gamma_i \ge |\tau_i[u]|, \;\; i = 1, 2, \dots, N-1. \tag{4.1}$$

Proof. It is an easy calculation (see details in [16, p. 6]) to verify that

$$\mathcal{L}^N\gamma_i \ge C_3\left(N^{-1} + [\max\{\varepsilon, h_{i+1}\}]^{-1}\bar{y}_i^N N^{-1}\right) =: \kappa_i,$$

where the constant C_3 can be made sufficiently large by choosing C_1 and C_2 large enough.

From Theorem 3.2, it is clear that

$$|\tau_i[u]| \le CN^{-1} \le \kappa_i \le \mathcal{L}^N\gamma_i, \quad i = J, J+1, \dots, N-1.$$

We next consider the two cases of Theorem 3.2 for $i \le J-1$. The first case, when $h_i > \varepsilon$, immediately yields $|\tau_i[u]| \le \kappa_i$ because $h_{i+1} \ge h_i$ for any i. Therefore, (4.1) is fulfilled in this case.

When $h_i \le \varepsilon$, we consider different values of the index i. First for $i \le J-3$, because of Remark 2.3 we have $h_{i+1} \le Ch_i$, so $h_{i+1} \le C\varepsilon$.

$$|\tau_i[u]| \le C\left(N^{-1} + \varepsilon^{-1}\bar{y}_i^N N^{-1}\right) \le \kappa_i \le \mathcal{L}^N\gamma_i \quad \text{for any } i \le J-3. \tag{4.2}$$

For the remaining cases, that is when $i = J-2, J-1$. If $h_{i+1} \le \varepsilon$, we have the same situation as above and estimate (4.2) is achieved for $i = J-2, J-1$. On the other hand when $h_{i+1} > \varepsilon$ for $i = J-2, J-1$, it imples $\max\{\varepsilon, h_{i+1}\} = h_{i+1}$ and $\varepsilon \le CN^{-1}$, we can show that

$$|\tau_i[u]| \le C\left(N^{-1} + h_{i+1}^{-1}\bar{y}_i^N N^{-1}\right).$$

Indeed, we modify the approach in (3.13), and note that $\sqrt{\varepsilon} < CN^{-1/2}$ because of $h_{i+1} > \varepsilon$ which gives $q - t_{J-2} \le N^{-1/2}$,

$$\begin{aligned}
P_i &\le C\hbar_i^{-1}e^{-\beta x_{i-1}/\varepsilon} \le Ch_{i+1}^{-1}e^{-\beta x_i/(2\varepsilon)}e^{-\beta x_i/(2\varepsilon)}, \quad (\text{since } h_i \le \varepsilon)\\
&\le Ch_{i+1}^{-1}\bar{y}_i^N\left(e^{-\beta x_{J-2}/2\varepsilon}\right) \le Ch_{i+1}^{-1}\bar{y}_i^N\left(e^{-a\beta q/(q-t_{J-2})}\right)\\
&\le Ch_{i+1}^{-1}\bar{y}_i^N N^{-1}.
\end{aligned}$$

The same argument applied for $R_i,\ Q_i$ gives

$$|\tau_i[y]| \le Ch_{i+1}^{-1}\bar{y}_i^N N^{-1} \le \kappa_i \le \mathcal{L}^N\gamma_i, \quad i = J-2, J-1. \tag{4.3}$$

This completes the proof. $\square$

THEOREM 4.2. *Let u be the solution of the continuous problem (1.1) and let w^N be the solution of the discrete problem (3.1) on the VB mesh. Then the following error estimate is satisfied:*

$$|u_i - w_i^N| \le CN^{-1}, \quad i = 0, 1, \dots, N.$$

Proof. We have $\gamma_i \ge 0 = (u - w^N)_i$ for $i = 0, N$, whereas (3.2) and Lemma 4.1 imply that $\mathcal{L}^N\gamma_i \ge \pm\mathcal{L}^N(u - w^N)_i$ for $i = 1, 2, \dots, N-1$. Then the discrete maximum principle gives $|u_i - w_i^N| \le \gamma_i$, $i = 0, 1, \dots, N$, and the assertion follows. $\square$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

5. Numerical results. The Bakhvalov mesh [3] and its generalization [22] were originally designed for reaction-diffusion problems. In particular, the VB mesh's numerical performance can be found in [22]. On the other hand, we now illustrate its optimal convergence result for the convection-diffusion problem (1.1) even with large values of ε. In order to verify the numerical rate of convergence of the upwind scheme on the VB mesh, we also calculate the convergence order ρ as a power of N^{-1},

$$\rho \approx \frac{\ln E_N - \ln E_{2N}}{\ln 2},$$

where $E_N = \max\limits_{0\le i\le N} |u_i - w_i^N|$. We choose $a = 2$ and $q = 1/2$ in all numerical experiments.

We consider the test problem taken from [13, p. 235],

$$-\varepsilon u'' - \left[x \sin\left(\frac{\pi}{4}x\right) + 1\right] u' + \left(2e^x + x^2\right) u = e^{x-1}, \quad u(0) = u(1) = 0. \tag{5.1}$$

Since we do not know the exact solution to this problem, we approximate the error E_N using the double-mesh principle (see, e.g., [5]). The results are presented in Table 5.1.

$-\log\varepsilon$	$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	
1	3.35e-03	1.82e-03	9.47e-04	4.84e-04	2.45e-04	1.23e-04	E_N
	0.88	0.94	0.97	0.98	0.99		ρ
2	6.21e-03	3.32e-03	1.72e-03	8.73e-04	4.40e-04	2.21e-04	
	0.90	0.95	0.98	0.99	0.99		
3	6.88e-03	3.61e-03	1.85e-03	9.38e-04	4.72e-04	2.37e-04	
	0.93	0.96	0.98	0.99	1.00		
4	7.00e-03	3.67e-03	1.88e-03	9.52e-04	4.79e-04	2.40e-04	
	0.93	0.97	0.98	0.99	1.00		
5	7.04e-03	3.69e-03	1.89e-03	9.56e-04	4.81e-04	2.41e-04	
	0.93	0.97	0.98	0.99	1.00		
6	7.05e-03	3.69e-03	1.89e-03	9.57e-04	4.81e-04	2.41e-04	
	0.93	0.97	0.98	0.99	1.00		
7	7.06e-03	3.70e-03	1.89e-03	9.57e-04	4.82e-04	2.42e-04	
	0.93	0.97	0.98	0.99	1.00		
8	7.06e-03	3.70e-03	1.89e-03	9.57e-04	4.82e-04	2.42e-04	
	0.93	0.97	0.98	0.99	1.00		

TABLE 5.1
The maximum pointwise error E_N and the convergence order ρ for the problem (5.1).

Table (5.1) clearly shows that the method converges uniformly in ε and that the order of convergence is 1. This order of ε-uniform convergence is optimal because the VB mesh, like other B-type meshes, does not suffer from $\ln N$-factors in the error like the piecewise-uniform S-type meshes do. The optimal order of ε-uniform convergence is a well-known property of B-type meshes (for instance, [11, Sec. 4.2.2] and [15] establish first-order convergence results, whereas second-order convergence can be found in [6, 7]).

6. Concluding remarks. We have analyzed the ε-uniform convergence of the upwind discretization of a linear singularly perturbed convection-diffusion problem on a Bakhvalov-type (B-type) mesh, where ε is the perturbation parameter. Our proof of ε-uniform convergence uses a barrier-function approach. So far, this proof method has only been applied to Shishkin-type (S-type) meshes and not to B-type meshes. The proof is an extension of the new barrier-function technique which we proposed in [16] and applied to a generalized Shishkin mesh. A new technique is needed because the generalized Shishkin mesh in [16] only retains the transition point, whereas the B-type mesh considered here does not have any of the features of S-type meshes. Otherwise, the classical barrier-function proof on S-type meshes primarily relies on the *a priori* defined transition point between fine and coarse parts of the mesh and, in the fine part, either on its uniformity, or on the smoothness of the functions generating the mesh in the layer region.

Our interest in B-type meshes can be justified by the fact that they do not have $\ln N$-factors in the error and are therefore numerically superior to the piecewise-uniform Shishkin meshes. However, our paper does not provide a unifying analysis for all B-type meshes, but only deals with one specific B-type mesh, the simplest one from [22]. A unifying analysis like this is possible when a hybrid stability-inequality (as opposed to the barrier-function estimate of the truncation error) is used (see, for example, [6, 7, 11]). The question, therefore, arises whether our technique can be extended to the original Bakhvalov mesh and its modifications in [6, 7] or [18, p. 120]. We believe that this is possible to do because all these meshes share many crucial properties, but some technical details may have to be different. We plan to work on generalizing our analysis so that it can be applied the whole class of B-type meshes from [22], which includes the original Bakhvalov mesh.

Although the mesh in [6] is an example showing that a B-type mesh does not have to be smooth in the layer region, this smoothness is almost a defining characteristic of B-type meshes. This is because of the mesh-generating functions used to create the points in the layer. The proof presented here, in its technical details, uses this smoothness, but, based on our result in [16], we do not feel this is essential for our barrier-function technique. There are also B-type meshes (see [25] for instance) which do not transition smoothly from the fine to the coarse part. Therefore, the smooth transition is also a property that probably could be eliminated but, again, some technicalities in the proof of ε-uniform convergence would very likely have to be different.

The barrier-function technique is limited to schemes that satisfy the discrete maximum principle. However, such schemes are natural for problems like (1.1), which satisfy the continuous maximum principle. Admittedly, the proof of ε-uniform convergence on B-meshes using barrier functions is not the simplest one (cf. the proof of Lemma 3.1), but the barrier-function technique is of interest because it can be extended to higher-dimensional problems. Proofs on the piece-wise uniform Shishkin mesh are simpler but the order of convergence is sub-optimal as a trade-off. On the other hand, barrier-function proofs of ε-uniform convergence on graded S-type meshes are essentially as involved as the proof presented here. This is why other proof techniques are usually used with S-type meshes, but they are often combined with the assumption that $\varepsilon \le N^{-1}$ (see [11, p. 12] and [12] for instance). Although this assumption is quite acceptable in practice, strictly speaking, it does not mean convergence uniform in ε. We point out that the proof presented here is valid for all values of ε.

As for higher-dimensional convection-diffusion problems, the barrier-function proofs of ε-uniform convergence has only been done on Shishkin-type meshes [9, 10]. Roos and Stynes [19] point out that the ε-uniform convergence proof for two-dimensional convection-diffusion problems on a Bakhvalov-type mesh is one of sixteen contemporary open questions in the numerical analysis of singularly perturbed differential equations. The present paper is a step in the direction of answering this question. The remaining task is to adapt and extend our barrier-function approach to 2D problems. This is our ongoing research project. Some results related to this are sketched in the appendix.

Acknowledgment. The authors would like to thank the referees for their careful reading and valuable corrections and suggestions. The work of Thái Anh Nhan was partially supported by the Faculty Development Program, Holy Names University.

REFERENCES

[1] V. B. Andreev, I. A. Savin, The uniform convergence with respect to a small parameter of A. A. Samarskii's monotone scheme and its modification, Comp. Math. Phys. 35 (1995), 581–591.
[2] V. B. Andreev, N. V. Kopteva, On the convergence, uniform with respect to a small parameter, of monotone three-point finite difference approximations, Differ. Equ. 34 (7) (1998), 921–929.
[3] N. S. Bakhvalov, The optimization of methods of solving boundary value problems with a boundary layer, USSR Comp. Math. Math. Phys. 9 (1969), 139–166.
[4] K. W. Chang, F. A. Howes, Nonlinear Singular Perturbation Phenomena: Theory and Application, Applied Mathematical Sciences, Vol. 56, Springer, New York, 1984.
[5] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, Robust Computational Techniques for Boundary Layers, Chapman & Hall/CRC, Boca Raton (2000).
[6] N. Kopteva, On the convergence, uniform with respect to the small parameter, of a scheme with central difference on refined grids, Comput. Math. Math. Phys. 39 (1999), 1594–1610.
[7] N. Kopteva, Uniform pointwise convergence of difference schemes for convection-diffusion problems on layer-adapted meshes, Computing, 66 (2001), 179–197.
[8] T. Linß, Layer-adapted meshes for convection-diffusion problems, Comput. Methods Appl. Mech. Eng. 192(9-10) (2003), 1061–1105.
[9] T. Linß, An upwind difference scheme on a novel Shishkin-type mesh for a linear convection-diffusion problem, J. Comput. Appl. Math. 110(1) (1999), 93–104.
[10] T. Linß, M. Stynes, A hybrid difference scheme on a Shishkin mesh for linear convection-diffusion problems. Appl. Numer. Math. 31(3) (1999), 255–270.
[11] T. Linß, Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems, Lecture Notes in Mathematics, Vol. 1985, Springer, Berlin, Heidelberg (2010).
[12] T. Linß, H.-G. Roos, R. Vulanović, Uniform pointwise convergence on Shishkin-type meshes for quasilinear convection-diffusion problems, SIAM J. Numer. Anal. 38 (2001), 897–912.
[13] T. A. Nhan, M. Stynes, R. Vulanović, Optimal uniform-convergence results for convection-diffusion problems in one dimension using preconditioning, J. Comput. Appl. Math. 338 (2018), 227–238, DOI: 10.1016/j.cam.2018.02.012.
[14] T. A. Nhan, R. Vulanović, Preconditioning and uniform convergence for convection-diffusion problems discretized on Shishkin-type meshes, Advances in Numerical Analysis, vol. 2016, Article ID 2161279.
[15] T. A. Nhan, R. Vulanović, Uniform convergence on a Bakhvalov-type mesh using preconditioning approach: Technical report, `arXiv: 1504.04283 math.NA` (2015).
[16] T. A. Nhan, R. Vulanović, A note on a generalized Shishkin-type mesh, Novi Sad J. Math, Vol. 48 No. 2 (2018), 141–150, DOI: 10.30755/NSJOM.07880.
[17] H.-G. Roos, T. Linß, Sufficient conditions for uniform convergence on layer-adapted grids, Computing, 63 (1999), 27–45.
[18] H.-G. Roos, M. Stynes, L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer Series in Computational Mathematics, Vol. 24, second ed., Springer, Berlin (2008).
[19] H.-G. Roos, M. Stynes, Some open questions in the numerical analysis of singularly perturbed differential equations, CMAM 15 (2015), 531–550.
[20] G. I. Shishkin, Grid approximation of singularly perturbed elliptic and parabolic equations (in Russian), Second Doctoral thesis, Keldysh Institute Moscow, 1990.
[21] M. Stynes, H.-G. Roos, The midpoint upwind scheme, Appl. Numer. Math. 23 (1997), 361–374.
[22] R. Vulanović, On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretization mesh, Univ. u Novom Sadu Zb. Rad. Prir. Mat. Fak. Ser. Mat. 13 (1983), 187–201.
[23] R. Vulanović, A higher-order scheme for quasilinear boundary value problems with two small parameters, Computing 67 (2001), 287–303.
[24] R. Vulanović, A priori meshes for singularly perturbed quasilinear two-point boundary value problems, IMA J. Numer. Anal. 21 (2001), 349–366.
[25] R. Vulanović, D. Herceg, N. Petrović, On the extrapolation for a singularly perturbed boundary value problem, Computing 36 (1986), 69–79.
[26] R. Vulanović, T. A. Nhan, Uniform convergence via preconditioning, Int. J. Numer. Anal. Model. Ser. B 5 (2014), 347–356.

Appendix: A sketch of the proof for the 2D convection-diffusion problem. We outline here the proof of ε-uniform convergence for the upwind discretization of the two-dimensional singularly perturbed convection-diffusion problem on the VB mesh. The problem, which corresponds to the one-dimensional case (1.1) with $c \equiv 0$ (assumed for simplicity), is given by

$$
\begin{aligned}
-\varepsilon \Delta u - b_1(x,y)u_x - b_2(x,y)u_y &= f(x,y) \quad \text{in} \quad \Omega = (0,1)^2, \\
u &= 0 \quad \text{on} \quad \Gamma = \partial\Omega,
\end{aligned}
\tag{6.1}
$$

where $b_1(x,y) \geq \beta_1 > 0$ and $b_2(x,y) \geq \beta_2 > 0$ for all $(x,y) \in \bar{\Omega}$. When the data satisfy the compatibility conditions (see, for instance, [10, Lemma 1]), the problem (6.1) has a classical solution $u \in C^{3,1}\left(\bar{\Omega}\right)$ and this solution can be decomposed as

$$
u = S + E_1 + E_2 + E_{12}. \tag{6.2}
$$

For details of the derivative estimates of the regular part S, the layer terms E_1 and E_2, and the corner component E_{12}, we refer the reader to [9, 10].

Let $\Omega^N = \{(x_i, y_j) : \ i,j = 0,\dots,N\}$ be the discretization mesh, where the points x_i and y_j form the respective VB meshes on the x- and y-axes. Let $h_{x,i} = x_i - x_{i-1},\ \hbar_{x,i} = (h_{x,i+1} + h_{x,i})/2$ and $h_{y,j} = y_j - y_{j-1},\ \hbar_{y,j} = (h_{y,j+1} + h_{y,j})/2$. We discretize the problem (6.1) by the standard upwind scheme as used in [10],

$$
\begin{aligned}
\mathcal{L}^N w_{ij}^N := \left(-\varepsilon(D_x'' + D_y'') - b_{1,ij}D_x^+ - b_{2,ij}D_y^+\right) w_{ij}^N &= f_{ij} \text{ on } \Omega^N \backslash \Gamma^N, \\
w_{ij}^N &= 0 \text{ on } \Gamma^N,
\end{aligned}
\tag{6.3}
$$

with

$$
D_x'' w_{ij}^N = \frac{1}{\hbar_{x,i}}\left(D_x^+ w_{ij}^N - D_x^- w_{ij}^N\right), \quad D_y'' w_{ij}^N = \frac{1}{\hbar_{y,j}}\left(D_y^+ w_{ij}^N - D_y^- w_{ij}^N\right),
$$

$$
D_x^- w_{ij}^N = \frac{w_{ij}^N - w_{i-1,j}^N}{h_{x,i}}, \quad D_x^+ w_{ij}^N = \frac{w_{i+1,j}^N - w_{i,j}^N}{h_{x,i+1}},
$$

$$
D_y^- w_{ij}^N = \frac{w_{ij}^N - w_{i,j-1}^N}{h_{y,j}}, \quad D_y^+ w_{ij}^N = \frac{w_{i,j+1}^N - w_{i,j}^N}{h_{y,j+1}}.
$$

Here, $\{w_{ij}^N\}$ is a mesh function on Ω^N, which represents the numerical solution, $w_{ij}^N \approx u(x_i, y_j)$. The numerical solution is decomposed analogously to (6.2),

$$
w_{ij}^N = S_{ij}^N + E_{1,ij}^N + E_{2,ij}^N + E_{12,ij}^N.
$$

We also split $\mathcal{L}^N$ into $\mathcal{L}_x^N + \mathcal{L}_y^N$, where

$$
\mathcal{L}_x^N w_{ij}^N = (-\varepsilon D_x'' - b_{1,ij}D_x^+)w_{ij}^N \ \text{ and } \ \mathcal{L}_y^N = (-\varepsilon D_y'' - b_{2,ij}D_y^+)w_{ij}^N.
$$

Next, we outline the truncation-error estimates. We use

$$
\begin{aligned}
\left|\mathcal{L}^N\left(u_{ij} - w_{ij}^N\right)\right| \leq \left|\mathcal{L}^N\left(S_{ij} - S_{ij}^N\right)\right| &+ \left|\mathcal{L}^N\left(E_{1,ij} - E_{1,ij}^N\right)\right| \\
&+ \left|\mathcal{L}^N\left(E_{2,ij} - E_{2,ij}^N\right)\right| + \left|\mathcal{L}^N\left(E_{12,ij} - E_{12,ij}^N\right)\right|
\end{aligned}
\tag{6.4}
$$

and separately bound each term of the right-hand side.

It is an easy calculation to show that

$$
\left|\mathcal{L}^N\left(S_{ij} - S_{ij}^N\right)\right| \leq CN^{-1}. \tag{6.5}
$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

For $\left|\mathcal{L}^N\left(E_{1,ij}-E_{1,ij}^N\right)\right|$, we follow the following key observation:

$$\left|\mathcal{L}^N\left(E_{1,ij}-E_{1,ij}^N\right)\right| \le \left|\mathcal{L}_x^N\left(E_{1,ij}-E_{1,ij}^N\right)\right| + \left|\mathcal{L}_y^N\left(E_{1,ij}-E_{1,ij}^N\right)\right|.$$

We can now imitate the truncation error analysis of Theorem 3.2 to get

$$\left|\mathcal{L}_x^N\left(E_{1,ij}-E_{1,ij}^N\right)\right| \le \begin{cases} C\left(N^{-1}+h_{x,i+1}^{-1}\bar{E}_{ij}^x N^{-1}\right) & \text{for } h_{x,i}>\varepsilon \text{ \& any } h_{y,j}, \\ C\left(N^{-1}+\varepsilon^{-1}\bar{E}_{ij}^x N^{-1}\right) & \text{for } h_{x,i}\le\varepsilon \text{ \& any } h_{y,j}, \end{cases}$$

where

$$\bar{E}_{ij}^x = \prod_{k=1}^{i}\left(1+\frac{\beta_1 h_{x,k}}{2\varepsilon}\right)^{-1} \quad \text{and} \quad \bar{E}_{ij}^y = \prod_{k=1}^{j}\left(1+\frac{\beta_2 h_{y,k}}{2\varepsilon}\right)^{-1}.$$

For $\left|\mathcal{L}_y^N\left(E_{1,ij}-E_{1,ij}^N\right)\right|$, we can easily show that, with arbitrary $h_{x,i}$,

$$\left|\mathcal{L}_y^N\left(E_{1,ij}-E_{1,ij}^N\right)\right| \le C\left(h_{y,j}+h_{y,j+1}\right) \le CN^{-1}.$$

where we used the property $h_{y,j}\le CN^{-1},\ j=1,\ldots,N$, of the VB mesh. Combining these bounds we get

$$\left|\mathcal{L}^N\left(E_{1,ij}-E_{1,ij}^N\right)\right| \le \begin{cases} C\left(N^{-1}+h_{x,i+1}^{-1}\bar{E}_{ij}^x N^{-1}\right) & \text{for } h_{x,i}>\varepsilon \text{ \& any } h_{y,j}, \\ C\left(N^{-1}+\varepsilon^{-1}\bar{E}_{ij}^x N^{-1}\right) & \text{for } h_{x,i}\le\varepsilon \text{ \& any } h_{y,j}. \end{cases} \tag{6.6}$$

Analogously,

$$\left|\mathcal{L}^N\left(E_{2,ij}-E_{2,ij}^N\right)\right| \le \begin{cases} C\left(N^{-1}+h_{y,j+1}^{-1}\bar{E}_{ij}^y N^{-1}\right) & \text{for } h_{y,j}>\varepsilon \text{ \& any } h_{x,i}, \\ C\left(N^{-1}+\varepsilon^{-1}\bar{E}_{ij}^y N^{-1}\right) & \text{for } h_{y,j}\le\varepsilon \text{ \& any } h_{x,i}. \end{cases} \tag{6.7}$$

Finally, for the corner component, we have

$$\left|\mathcal{L}^N\left(E_{12,ij}-E_{12,ij}^N\right)\right| \le CN^{-1}\begin{cases} 1+h_{x,i+1}^{-1}\bar{E}_{ij}^x+h_{y,j+1}^{-1}\bar{E}_{ij}^y, & h_{x,i}>\varepsilon \text{ \& } h_{y,j}>\varepsilon, \\ 1+h_{x,i+1}^{-1}\bar{E}_{ij}^x+\varepsilon^{-1}\bar{E}_{ij}^y, & h_{x,i}>\varepsilon \text{ \& } h_{y,j}\le\varepsilon, \\ 1+\varepsilon^{-1}\bar{E}_{ij}^x+h_{y,j+1}^{-1}\bar{E}_{ij}^y, & h_{x,i}\le\varepsilon \text{ \& } h_{y,j}>\varepsilon, \\ 1+\varepsilon^{-1}\left[\bar{E}_{ij}^x+\bar{E}_{ij}^y\right], & h_{x,i}\le\varepsilon \text{ \& } h_{y,j}\le\varepsilon. \end{cases} \tag{6.8}$$

From (6.5), (6.6), (6.7), and (6.8), we get

$$\left|\mathcal{L}^N\left(u_{ij}-w_{ij}^N\right)\right| \le CN^{-1}\begin{cases} 1+h_{x,i+1}^{-1}\bar{E}_{ij}^x+h_{y,j+1}^{-1}\bar{E}_{ij}^y, & h_{x,i}>\varepsilon \text{ \& } h_{y,j}>\varepsilon, \\ 1+h_{x,i+1}^{-1}\bar{E}_{ij}^x+\varepsilon^{-1}\bar{E}_{ij}^y, & h_{x,i}>\varepsilon \text{ \& } h_{y,j}\le\varepsilon, \\ 1+\varepsilon^{-1}\bar{E}_{ij}^x+h_{y,j+1}^{-1}\bar{E}_{ij}^y, & h_{x,i}\le\varepsilon \text{ \& } h_{y,j}>\varepsilon, \\ 1+\varepsilon^{-1}\left[\bar{E}_{ij}^x+\bar{E}_{ij}^y\right], & h_{x,i}\le\varepsilon \text{ \& } h_{y,j}\le\varepsilon. \end{cases} \tag{6.9}$$

ETNA
Kent State University and
Johann Radon Institute (RICAM)

We now proceed to form a barrier function, similar to the one used in the one-dimensional case,

$$\gamma_{ij} = \tilde{C}\left[\gamma_{ij}^s + \gamma_{ij}^x + \gamma_{ij}^y\right],$$

where $\tilde{C}$ is a sufficiently large constant independent of N and ε, and

$$\gamma_{ij}^s = N^{-1}\left((1-x_i) + (1-y_j)\right), \;\; \gamma_{ij}^x = \bar{E}_{ij}^x N^{-1}, \;\; \text{and} \;\; \gamma_{ij}^y = \bar{E}_{ij}^y N^{-1}.$$

By imitating Lemma 4.1, we can show that $\mathcal{L}^N\gamma_{ij} \geq \left|\mathcal{L}^N\left(u_{ij} - w_{ij}^N\right)\right|$ for all cases described in (6.9) depending on the values of the indices i and j. Using the fact that the discrete operator $\mathcal{L}^N$ defined in (6.3) satisfies the discrete maximum principle, we arrive at the main result: *The error estimate of the upwind scheme discretizing the problem* (6.1) *on the Vulanović-Bakhvalov mesh satisfies*

$$\left|u_{ij} - w_{ij}^N\right| \leq CN^{-1}, \quad 0 \leq i,\; j \leq N.$$