The Unicity Theorem For Meromorphic Maps Of A Complete Kähler Manifold Into $P^N(\mathbb{C})$

Thai Anh Nhan
Department of Mathematics, Can Tho University,
Can Tho city, Vietnam

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Introduction

The uniqueness problem for meromorphic mappings was first studied by R. Nevanlinna in 1926. He proved the following unicity theorem for meromorphic functions on C:

Theorem 0.1. *Let ϕ, ψ be nonconstant meromorphic functions on $\mathbb{C}$. If there exist five distinct values $a_1, \ldots, a_5$ such that $\phi^{-1}(ai) = \psi^{-1}(ai), (1 \leq i \leq 5)$, then $\phi \equiv \psi$.*

During recent decades, there were many generalized results of Nevanlinna's theorem in the case where meromorphic maps of $\mathbb{C}^n$ into a complex projective space $P^n(\mathbb{C})$. In 1975, for instance, H. Fujimoto [4] showed that if two meromorphic maps f and g of C^n into $P^n(\mathbb{C})$ have the same inverse images (regarding multiplicities) for $(3N+2)$ hyperplanes in general position, then $f \equiv g$. Since then, this topic has been studied strongly and continuously with the results of H. Fujimoto, W. Stoll, L. Smiley, M. Ru, D. D. Thai, Z. Ye and so on.

Results of many mathematicians on this subject regularly published for last 30 years have showed the fascination of the uniqueness problem for meromorphic mappings. For this reason, we chose the thesis: The Unicity Theorem For Meromorphic Maps Of A Complete Kähler Manifold Into $P^N(\mathbb{C})$, in order to initially learn about Value Distribution Theory and its important application- Uniqueness problem.

Up to now, unicity problem has been solved almost based on the results of Theory of Value Distribution- A theory remains many interesting open questions in itself and it is being applied in different mathematical disciplines such as Diophantine approximation, hyperbolic complex analysis, complex dynamics, differential equations,...

It is said that investigating the uniqueness problems for meromorphic maps needs both aspects: constructing Value Distribution Theory (especially in establishing the Second Main Theorem) and studying its applications. Consequently, this thesis has two chapters:
Chapter 1: Value Distribution Theory in several complex variables.

Chapter 2: The unicity theorem for meromorphic maps of a complete Kähle manifold into $P^N(\mathbb{C})$.

The purpose of Chapter one based on H. Fujimoto's papers [6, 7, 8] is to establish the Second Main Theorem, a key theorem in Nevanlinna Theory, for meromorphic maps of $\mathbb{C}^n$ into $P^N(\mathbb{C})$. The lemma of the logarithmic derivative- the fundamental lemma in proving the Second Main Theorem is explicitly solved in this Chapter. Chapter one is also considered the preliminaries for the proofs in Chapter two.

The main content of dissertation is in Chapter two. We tried to explicitly present and concretely prove H. Fujimoto's results in [9], study meromorphic maps of an n-dimensional complete Kähler manifold M into $P^N(\mathbb{C})$ and give a new type of the unicity theorem in the case where the universal covering of M is biholomorphic to the ball in $\mathbb{C}^n$ and meromorphic maps satisfy a (C_ρ) condition. That means letting M is an n-dimensional connected Kähler manifold with Kähler form ! and f is a meromorphic map of M into $P^N(\mathbb{C})$. For $\rho \geq 0$ we say that f satisfies the condition $C_\rho)$ if there exists a non-zero bounded continuous real-valued function h on M such that

$$\rho\Omega_f + dd^c \log h^2 \geq Ric\omega$$

where Ω_f denotes the pull-back of the Fubini-Study metric form on $P^N(\mathbb{C})$ and $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$.

One big drawback to people who have just started to read Nevanlinna's Theory of Value Distribution is that this theory concerns many profound results in complex analysis, meromorphic maps, analytic sets, complex geometry, etc. The author could not get over these obstacles unless there were wholehearted instructions of his supervisor. On this occasion, the author would like to express a deep gratitude to Prof. Do Duc Thai for his enthusiastic supervising and useful lectures he gave. The author would also like to thank professors and lecturers at Faculty of Mathematics, Hanoi National University of Education, Geometry Division, Seminars on complex geometry, Seminars on Value Distribution Theory. The 2 author also thank referees who spent their precious time for reading and giving helpful advices for his thesis. At last, but most important the author is deeply grateful to his family. This is a huge motivation both in morale and financial supports in helping author to finish Master's course in Hanoi.

Chapter 1

Value Distribution Theory in several complex variables

1.1 Preliminaries

Let M be a n-dimensional complex manifold and let $f : M \to P^N(\mathbb{C})$ be a meromorphic map. Let $p \in M$ and we denote the germ of all meromorphic functions at p by $\mathcal{M}_p$. Let U be a neighborhood of local holomorphic coordinates at p such that U is a Cousin domain II. Then f has a reduced representation on U, i.e $f = (f_1 : \ldots : f_{N+1}$ such that each f_i is a holomorphic function on U, with $\mathsf{f} := (f_1, \ldots, f_{N+1}) \not\equiv 0$ and $f(z) = (f_1(z) : \ldots : f_{N+1}(z)$. In addition, the set $\{z \in U : f_i(z) = 0, 1 \le i \le N+1\}$ has co-dimension which is higher than 1. With $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i is a non-positive integer, we denote

$$D^\alpha \mathsf{f} = \left(\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} f_1, \ldots, \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} f_{N+1} \right) \in \mathcal{M}_p^{N+1},$$

where $D^0\mathsf{f} = (f_1, \ldots, f_{N+1})$. For each $k \ge 0$, we denote $\mathcal{F}_p^k$ a $\mathcal{M}_p$-submodule of $\mathcal{M}_p^{N+1}$ generated by $\{D^\alpha \mathsf{f} : |\alpha| \leqq k\}$ and let $\mathcal{F}_p^{-1} = \{0\}$.

Proposition 1.1. *The set $\mathcal{F}_p^k$ does not depend on the chooses of local holomorphic coordinate systems and reduced representation $f = (f_1 : \ldots : f_{N+1})$.*

Proof. We prove by induction on k. It is easy to see that the proposition holds for $k = 0$ as we have a biholomorphic between two local coordinate systems. Suppose that the proposition holds for all $|\alpha| \le k$. Let $u = (u_1, \ldots, u_n)$ be other local coordinates. We use the following notation

$${}^uD_i = \frac{\partial}{\partial u_i}, \quad {}^uD^\alpha = \frac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1} \ldots \partial u_n^{\alpha_n}},$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$. We take an arbitrary α, with $|\alpha| = k+1$. We write $D^\alpha = D_i D^{\alpha'}$, where $1 \leq i \leq n$ and $|\alpha'| = k$. By the induction assumption, we can write

$$^u D^{\alpha'} \mathsf{f} = \sum_{|\beta| \leqq k} h_\beta D^\beta \mathsf{f},$$

where $h_\beta \in \mathcal{M}_p$. So we have

$$\begin{aligned} ^u D^\alpha \mathsf{f} &= {}^u D_i ({}^u D^{\alpha'} \mathsf{f}) \\ &= {}^u D_i \left(\sum_{|\beta| \leqq k} h_\beta D^\beta \mathsf{f} \right) \\ &= \sum_{|\beta| \leqq k} \left({}^u D_i h_\beta D^\beta \mathsf{f} + \sum_{j=1}^n h_\beta \frac{\partial z_j}{\partial u_i} D_i D^\beta \mathsf{f} \right) \in \mathcal{F}_p^{k+1} \end{aligned}$$

This implies that $\mathcal{F}_p^{k+1}$ does not depend on choosing local holomorphic coordinates.

Now we consider another reduced representation of $f = (\tilde{f}_1 : \ldots : \tilde{f}_{N+1})$ and let $\tilde{\mathsf{f}} := (\tilde{f}_1, \ldots, \tilde{f}_{N+1})$. Then $h = \frac{\tilde{f}_i}{f_i} (i = 1, 2 \ldots, N+1)$ are nowhere vanishing holomorphic functions. Let $D^\alpha = D_i D^{\alpha'}$, with $|\alpha'| = k$. By the induction assumption we can write

$$D^{\alpha'} \tilde{\mathsf{f}} = \sum_{|\beta| \leqq k} g_\beta D^\beta \mathsf{f},$$

where $g_\beta \in \mathcal{M}_p$ with $g_\beta \in \mathcal{M}_p$. Therefore, we obtain

$$D^\alpha \tilde{\mathsf{f}} = D_i D^{\alpha'} \tilde{\mathsf{f}} = D_i (\sum_{|\beta| \leqq k} g_\beta D^\beta \mathsf{f}) = \sum_{|\beta| \leq k} D_i g_\beta D^\beta \mathsf{f} + \sum_{|\beta| \leqq k} g_\beta D_i D^\beta \mathsf{f} \in \mathcal{F}_p^{k+1}.$$

This means $\mathcal{F}_p^{k+1}$ does not depend on choosing reduced representation of f □

Definition 1.1. *We define a kth rank of f as follows*

$$r_f(k) := rank_{\mathcal{M}_p} \mathcal{F}_p^k - rank_{\mathcal{M}_p} \mathcal{F}_p^{k-1}.$$

Definition 1.2. *A total rank of f and a total rank of Jacobi matrix of f are defined as*

$$r_f := \sum_{k \geqq 0} r_f(k) - 1,$$

and

$$l_f := \sum_{k \geqq 0} k r_f(k)$$

respectively.

Proposition 1.2. *1. A meromorphic map $f : M \to P^N(\mathbb{C})$ is non degenerate, i.e. $f(M)$ does not contain in any hyperplane of $P^N(\mathbb{C})$, iff $r_f = N$.*

2. $l_f \leqq \dfrac{N(N+1)}{2}$ for all meromorphic maps to $P^N(\mathbb{C})$.

Proof. First of all, we prove (1). Assume that f is degenerate, i.e. $f_1, \ldots, f_{N+1}$ are linear dependent on $\mathbb{C}$:

$$a^1 f_1 + \ldots + a^{N+1} f_{N+1} \equiv 0$$

where $(a^1, \ldots, a^{N+1}) \neq (0, \ldots, 0)$. So, we have:

$$a^1 D^\alpha f_1 + \ldots + a^{N+1} D^\alpha f_{N+1} \equiv 0, \quad \forall \alpha$$

$$\Longrightarrow \mathrm{rank}_{\mathcal{M}_p} \mathcal{F}_p^k = \mathrm{rank}_{\mathcal{M}_p} (D^\alpha \mathsf{f} : |\alpha| \leqq k) < N+1,$$

for all $k = 1, 2, \ldots$. Therefore, we get $r_f < N$.

Conversely, suppose that $r_f < N \Rightarrow \max_{k \geqq 0} \mathrm{rank}_{\mathcal{M}_p} \mathcal{F}_p^k < N+1$. Then, there exist $(\varphi_1, \ldots, \varphi_{N+1}) \neq (0, \ldots, 0)$ in $\mathcal{M}_p^{N+1}$ such that:

$$\varphi_1 D^\alpha f_1 + \ldots + \varphi_{N+1} D^\alpha f_{N+1} \equiv 0,$$

for all α. Taking $q \in M$ close enough to p such that $\varphi_1, \ldots, \varphi_{N+1}$ are holomorphic in a neighborhood of q and $(\varphi_1(q), \ldots, \varphi_{N+1}(q)) \neq (0, \ldots, 0)$. Let

$$\psi(z) = \varphi_1(q) f_1(z) + \ldots + \varphi_{N+1}(q) f_{N+1}(z)$$

on U. Then, we have

$$(D^\alpha \psi)(q) = \varphi_1(q) D^\alpha f_1(q) + \ldots + \varphi_{N+1}(q) D^\alpha f_{N+1}(q) = 0$$

for all α. By the identity theorem , we have $\psi \equiv 0$. This means that f is degenerate. □

Remark 1.1. *By this proof we can see that f is non degenerate iff rank$_{\mathcal{M}_p} \mathcal{F}_p^{k_0} = N+1$ with for some positive integer k_0.*

To prove (2), first we need to prove the following lemma

Lemma 1.1. *Let $\mathcal{F}_p^{k+1} = \mathcal{F}_p^k$ for some positive integer k. Then, we have $\bigcup_{k\geqq 0}\mathcal{F}_p^k = \mathcal{F}_p^k$.*

Proof. Suppose that we have $\mathcal{F}_p^{k+1} = \mathcal{F}_p^k$, we will prove $\mathcal{F}_p^{k'} \subset \mathcal{F}_p^k$ for all $k' > k$ by induction on k'. It is easy to see the case where $k' = k+1$. Suppose that $\mathcal{F}_p^{k'} \subset \mathcal{F}_p^k$ with $k' > k$. Taking α with $|\alpha| = k'+1$ and we write $D^\alpha = D_i D^{\alpha'}$ for some i and α' with $|\alpha'| = k'$. By the induction assumption we can write

$$D^{\alpha'}\mathsf{f} = \sum_{|\beta|\leqq k} \varphi_{\alpha'\beta} D^\beta \mathsf{f},$$

where $\varphi_{\alpha'\beta} \in \mathcal{M}_p$. We have,

$$D^\alpha \mathsf{f} = D_i D^{\alpha'}\mathsf{f} = \sum_{|\beta|\leqq k} D_i\varphi_{\alpha'\beta} D^\beta \mathsf{f} + \sum_{|\beta|\leqq k} \varphi_{\alpha'\beta} D_i D^\beta \mathsf{f} \in \mathcal{F}_p^{k'} \subset \mathcal{F}_p^k.$$

This proves our statement. □

Now we give the proof of (2).

Proof. We have

$$\mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^k \leqq N+1, \forall k = 1, 2, \ldots.$$

So, there is the smallest positive integer k_1 such that

$$1 = \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^0 < \ldots < \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1-1} < \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1} < N+1$$

and $\mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1} = \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1+1}$. Therefore,

$$\begin{aligned}
l_f = \sum_{k\geqq 0} k r_f(k) &= \sum_{k=1}^{k_1} k(\mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1} - \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1-1}) \\
&= k_1 \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1} - (\mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^0 + \ldots + \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^{k_1-1}) \\
&\leqq k_1(N+1) - (1 + \ldots + k_1) \\
&= \frac{N(N+1) - (N-k_1)^2 - (N-k_1)}{2} \leqq \frac{N(N+1)}{2}.
\end{aligned}$$

□

Now we consider a meromorphic map f from $B(R_0) := \{z \in \mathbb{C}^n : \|z\| < R_0\}, (0 < R_0 \leqq +\infty)$ into $P^N(\mathbb{C})$, where $\|z\| = (\sum_{i=1}^n \|z_i\|^2)^{\frac{1}{2}}$ with $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We use the following convention: $B(\infty) = \mathbb{C}^n$. Taking a reduced representation $f = (f_1 : \ldots : f_{N+1})$ in $B(R_0)$, we denote

$$\|f\| = (|f_1|^2 + \ldots + |f_{N+1}|^2)^{\frac{1}{2}}.$$

By the definition, a pullback of Fubini metric form of f is given by

$$\Omega_f := dd^c \log \|f\|^2.$$

We denote $\upsilon_l = (dd^c\|z\|^2)^l, \sigma_n = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}$ and $S(r) = \{z \in \mathbb{C}^n : \|z\| = r\}$.

Definition 1.3. *A characteristic function f is defined as follows*

$$T_f(r, r_0) = \int_{S(r)} \log \|f\|\sigma_n - \int_{S(r_0)} \log \|f\|\sigma_n.$$

Let ϕ be a non zero meromorphic function on $B(R_0)$. we can consider ϕ as a meromorphic function into $P^1(\mathbb{C})$. For each $a \in P^1(\mathbb{C})$, we denote the number of multiplicities of zero of $\phi - a$ at a point $z \in B(R_0)$ by $\nu_\phi^a(z)$. Taking

$$n_\phi^a(r) = \begin{cases} \frac{1}{r^{2n-2}} \int_{\{\phi=a\}\cap B(r)} \nu_\phi^a \upsilon_{n-1} & \text{if } n > 1 \\ \sum_{z\in B(r)} \nu_\phi^a(z) & \text{if } n = 1 \end{cases},$$

and determine a counting function of a by

$$N_\phi^a(r, r_0) = \int_{r_0}^{r} \frac{n_\phi^a(t)}{t} dt \quad (0 < r_0 < r < R_0).$$

So we obtain Jensen's formula:

$$\int_{S(r)} \log |\phi|\sigma_n - \int_{S(r_0)} \log |\phi|\sigma_n = N_\phi^0(r, r_0) - N_\phi^\infty(r, r_0). \tag{1.1}$$

Let $f : B(R_0) \to P^N(\mathbb{C})$ be a non-degenerate meromorphic function with a reduced representation $f = (f_1 : \ldots : f_{N+1})$ and let $\mathsf{f} = (f_1, \ldots, f_{N+1})$.

Definition 1.4. *Let $\alpha^i = (\alpha_1^i, \ldots, \alpha_n^i)$ $(1 \leqq i \leqq N+1)$ be $N+1$ multi-indices . A generalized Wronskian of f (or of f) is defined as*

$$W_{\alpha^1\ldots\alpha^{N+1}}(f) \equiv W_{\alpha^1\ldots\alpha^{N+1}}(\mathsf{f}) := \det(D^{\alpha^i}\mathsf{f} : 1 \leqq i \leqq N + 1).$$

Definition 1.5. *We say that $\{\alpha^1, \ldots, \alpha^{N+1}\}$ $(\alpha^i = (\alpha_1^i, \ldots, \alpha_n^i))$ is acceptable of f (or f) if for each $k \geqq 0$ $\{D^{\alpha^1}\mathsf{f}, \ldots, D^{\alpha^{l(k)}}\mathsf{f}\}$ is a base of $\mathcal{M}_p$-module $\mathcal{F}_p^k$, where $p \in M$ and $l(k) = rank_{\mathcal{M}_p}\mathcal{F}_p^k$.*

Proposition 1.3. *Let f be non degenerate and let $\{\alpha^1, \ldots, \alpha^{N+1}\}$ be acceptable of f. Then, we have*

$$W_{\alpha^1\ldots\alpha^{N+1}}(g\mathsf{f}) = g^{N+1}W_{\alpha^1\ldots\alpha^{N+1}}(\mathsf{f})$$

for abitrary non-zero holomophic map g, where $g\mathsf{f} = (gf_1, \ldots, gf_{N+1})$.

Proof. We prove by induction on $|\alpha|$, for each α we can write

$$D^\alpha(g\mathsf{f}) = gD^\alpha\mathsf{f} + \sum_{|\beta|<|\alpha|} a_{\alpha\beta}D^{\alpha-\beta}gD^\beta\mathsf{f}, \tag{1.2}$$

for some constants $a_{\alpha\beta}$. We replace each $D^{\alpha^i}(g\mathsf{f})$ in

$$W_{\alpha^1\dots\alpha^{N+1}}(g\mathsf{f}) = \det(D^{\alpha^i}(g\mathsf{f}) : 1 \leqq i \leqq N+1)$$

by the right hand side of equation (1.2) with $\alpha = \alpha^i$ and repeat with an addition of a multiple of another line we obtain

$$\det(D^{\alpha^i}(g\mathsf{f})) = \det(gD^{\alpha^i}\mathsf{f}).$$

This proves our statement. □

1.2 The lemma on logarithmic derivative

In this section, we prove the Lemma on logarithmic derivative [8].

Let $\varphi(z_1,\dots,z_n)$ be a non-zero meromorphic function which on $B(R_0)(0 < R_0 \leqq +\infty)$. Let $\alpha = (\alpha_1,\dots,\alpha_n)$ be a multi-index and let $z = (z_1,\dots,z_n) \in \mathbb{C}^n$. We denote $z^\alpha := z_1^{\alpha_1}\dots z_n^{\alpha_n}$ and $D^\alpha\varphi = D_1^{\alpha_1}\dots D_n^{\alpha_n}\varphi$, where $D_i\varphi = \dfrac{\partial}{\partial z_i}\varphi$. Then, we obtain the following theorem.

Theorem 1.1 (The lemma on logarithmic derivative)**.** *Let $\alpha = (\alpha_1,\dots,\alpha_n) \neq (0,\dots,0), 0 < r_0 < R_0$. Taking positive integers p, p' such that $0 < p|\alpha| < p' < 1$. Then, for $r_0 < r < R < R_0$, we have*

$$\int_{S(r)} \left| z^\alpha \left(\frac{D^\alpha\varphi}{\varphi}(z)\right)\right|^p \sigma_n(z) \leqq \left(\frac{R^{2n-1}}{R-r}T_\varphi(R,r_0)\right)^{p'},$$

where K is a constant which does not depend on r and R..

Corollary 1.1. *Let $\alpha = (\alpha_1,\dots,\alpha_n) \neq (0,\dots,0)$ and $0 < r_0 < R_0$. Then, for $r_0 < r < R < R_0$, we have*

$$\int_{S(r)} \log^+ |(D^\alpha\varphi/\varphi)(z)|\sigma_n(z) \leqq K\log^+\left(\frac{R^{2n-1}}{R-r}T_\varphi(R,r_0)\right).$$

To prove Corollary 1.1 using Theorem 1.1, we use the following result by Biancofiore and Stoll:

Let $h \geqq 0$ be an integrable function on $S(r)$. Then, we have

$$\int_{S(r)} \log^+ h\sigma_n \leqq \log^+ \int_{S(r)} h\sigma_n + \log 2. \tag{1.3}$$

To apply Theorem 1.1, we take p, p' such that $0 < p|\alpha| < p' < 1$. By (1.2) and Theorem 1.1, we have

$$\begin{aligned}\int_{S(r)} \log^+ |D^\alpha \varphi/\varphi|\sigma_n &\leqq \frac{1}{p}\int_{S(r)} \log^+ |z^\alpha (D^\alpha\varphi/\varphi)(z)|^p \sigma_n(z) + K \\ &\leqq \frac{1}{p}\log^+ \int_{S(r)} |z^\alpha (D^\alpha\varphi/\varphi)(z)|^p \sigma_n(z) + K \\ &\leqq K \log^+ \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right).\end{aligned}$$

Therefore, we obtain Corollary 1.1.

For the following part , we use K to denote a constant which does not depend on r with $r_0 < r < R_0$ even if this constant is replaced by a new one.

Next, we will prove Theorem 1.1. First of all, we prove the following lemma

Lemma 1.2. *Let φ_1, φ_2 be non-zero mermorphic functions on $B(R_0)$. Then, we have*

$$T_{\varphi_1\varphi_2}(r, r_0) \leqq T_{\varphi_1}(r, r_0) + T_{\varphi_2}(r, r_0) + K.$$

Proof. We take reduced representative points $\varphi_i = (g_i : h_i)$ $(i = 1, 2)$, and $\varphi_1\varphi_2 = (g_3 : h_3)$. Then, $k := g_1 g_2/g_3 = h_1 h_2/h_3$ are holomorphic. Since,

$$\begin{aligned}&(|g_1|^2|g_2|^2 + |h_1|^2|h_2|^2) \leqq (|g_1|^2 + |g_2|^2)(|h_1|^2 + |h_2|^2) \\ &\Rightarrow (|k|^2|g_3|^2 + |k|^2|h_3|^2) \leqq (|g_1|^2 + |g_2|^2)(|h_1|^2 + |h_2|^2) \\ &\Rightarrow |k|(|g_3|^2 + |h_3|^2)^{1/2} \leqq (|g_1|^2 + |g_2|^2)^{1/2}(|h_1|^2 + |h_2|^2)^{1/2} \\ &\Rightarrow \log(|g_3|^2 + |h_3|^2)^{1/2} + \log|k| \leqq \log(|g_1|^2 + |h_1|^2)^{1/2} + \log(|g_2|^2 + |h_2|^2)^{1/2},\end{aligned}$$

we have

$$\int_{S(r)} \log\|\varphi_1\varphi_2\|\sigma_n \leqq \int_{S(r)} \log\|\varphi_1\|\sigma_n + \int_{S(r)} \log\|\varphi_2\|\sigma_n - N_k(r, r_0) + K.$$

Therefore, we get

$$\int_{S(r)} \log\|\varphi_1\varphi_2\|\sigma_n \leqq \int_{S(r)} \log\|\varphi_1\|\sigma_n + \int_{S(r)} \log\|\varphi_2\|\sigma_n + K,$$

that means

$$T_{\varphi_1\varphi_2}(r, r_0) \leqq T_{\varphi_1}(r, r_0) + T_{\varphi_2}(r, r_0) + K.$$

This prove our lemma □

Lemma 1.3. *Let φ be a non-zero meromorphic function on $B(R_0)$ with a reduced representative $\varphi = g/h(= (g : h))$. Then, we have*

$$\int_{S(r)} |\log|\varphi||\sigma_n \leqq 2T_\varphi(r, r_0) + K.$$

Proof. Since

$$\begin{aligned} |\log|\varphi|| &= \left|\log \frac{|g|}{|h|}\right| = |\log|g| - \log|h|| \\ &= 2\max(\log|g|, \log|h|) - \log|g| - \log|h| \\ &\leqq 2\log\|\varphi\| - \log|g| - \log|h|, \end{aligned}$$

we have

$$\begin{aligned} \int_{S(r)} |\log|\varphi||\sigma_n &\leqq 2\int_{S(r)} \log\|\varphi\|\sigma_n - \int_{S(r)} \log|g|\sigma_n - \int_{S(r)} \log|h|\sigma_n \\ &\leqq 2T_\varphi(r, r_0) - N_g(r, r_0) - N_h(r, r_0) + K \\ &\leqq 2T_\varphi(r, r_0) + K. \end{aligned}$$

□

Lemma 1.4. $N_\varphi^a(r, r_0) \leqq T_\varphi(r, r_0) + K$ *for all* $a \in P^1(\mathbb{C})$.

Proof. Let $a = (a_1 : a_2) \in P^1(\mathbb{C})$, we have

$$\begin{aligned} N_\varphi^a(r, r_0) &= \int_{S(r)} \log|a_2 g - a_1 h|\sigma_n - \int_{S(r_0)} \log|a_2 g - a_1 h|\sigma_n \\ &\leqq \int_{S(r)} \log\|\varphi\|\sigma_n + K \\ &\leqq T_\varphi(r, r_0) + K. \end{aligned}$$

□

Lemma 1.5.

$$\left|T_\varphi(r, r_0) - \left(\int_{S(r)} \log^+|\varphi|\sigma_n + N_\varphi^\infty(r, r_0)\right)\right| \leqq K,$$

where $\log^+ = \max(\log x, 0)$ *with* $x \geqq 0$.

Proof. We have

$$\begin{aligned}\log\|\varphi\| &= \log(|g|^2+|h|^2)^{1/2}\\ &\leqq \log(2\max(|g|,|h|))\\ &= \log^+\left|\frac{g}{h}\right| + \log|h| + \log 2\end{aligned}$$

$$\Rightarrow \int_{S(r)} \log\|\varphi\|\sigma_n \leqq \int_{S(r)} \log^+|\varphi|\sigma_n + \int_{S(r)} \log|h|\sigma_n + K,$$

therefore

$$T_\varphi(r,r_0) - \left(\int_{S(r)} \log^+|\varphi|\sigma_n + N_\varphi^\infty(r,r_0)\right) \leqq K.$$

On the other hand, we have

$$\begin{aligned}\log^+|\varphi| + \log|h| + \log 2 &= \log(2\max(|g|,|h|))\\ &= \log(\max(|g|,|h|)) + \log 2\\ &\leqq \log(|g|^2+|h|^2)^{1/2} + K\\ &= \log\|\varphi\| + K,\end{aligned}$$

thus

$$\left(\int_{S(r)} \log^+|\varphi|\sigma_n + N_\varphi^\infty(r,r_0)\right) - T_\varphi(r,r_0) \leqq K.$$

This proves Lemma 1.5 □

Lemma 1.6. *Let* $0 < r_0 < r < R < R_0$ *and let* $\rho := (r+R)/2$*. Then, we have*

$$n_{\nu_\varphi^a}(\rho) \leqq \frac{2R}{R-r}(T_\varphi(R,r_0)+K).$$

Proof. By definition, we have

$$N_\varphi^a(R,r_0) = \int_{r_0}^R n_{\nu_\varphi^a}(t)\frac{dt}{t} \geqq \int_\rho^R n_{\nu_\varphi^a}(t)\frac{dt}{t} \geqq n_{\nu_\varphi^a}(\rho)\frac{R-\rho}{R},$$

so we obtain

$$n_{\nu_\varphi^a}(\rho) \leqq \frac{R}{R-\rho}N_\varphi^a(R,r_0) \leqq \frac{2R}{R-r}(T_\varphi(R,r_0)+K) \quad \text{(by Lemma 1.4).}$$

□

Lemma 1.7. *Let $r > 0$ and $0 < p < 1$. For all $a \in \mathbb{C}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{r^p}{|re^{i\theta}-a|^p}d\theta \leqq \frac{2-p}{2(1-p)}.$$

Proof. Without loss of generation, we can assume that a is a real positive number. If $|\theta| \leqq \frac{\pi}{2}$, we have

$$|re^{i\theta}-a| \geqq r|\sin\theta| \geqq \frac{2}{\pi}$$

and if $\pi/2 < |\theta| \leqq \pi$, we get $|re^{i\theta}-a| \geqq r$. Therefore, we obtain

$$\begin{aligned}\int_0^{2\pi}\frac{r^p}{|re^{i\theta}-a|^p}d\theta &\leqq 2\int_0^{\pi/2}\left(\frac{\pi}{2\theta}\right)^p d\theta + 2\int_{\pi/2}^{\pi}d\theta\\ &\leqq \frac{2^{1-p}\pi^p}{1-p}\left(\frac{\pi}{2}\right)^{1-p}+\pi\\ &= \frac{\pi(2-p)}{1-p}.\end{aligned}$$

This implies

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{r^p}{|re^{i\theta}-a|^p}d\theta \leqq \frac{2-p}{2(1-p)}.$$

□

Proposition 1.4 ([11])**.** *Let φ be a non-zero meromorphic function on $\{u \in \mathbb{C} : |u| < R_0\}$ and let $0 < r < R < R_0$. We take $z \in \mathbb{C}$ such that $|z| = r$ and $\varphi(z) \neq 0, \infty$. Then, we have*

$$\log|\varphi(z)| = \frac{1}{2\pi}\int_0^{2\pi}\log|\varphi(Re^{i\phi})|Re\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right)d\phi + \sum_{|u|\leqq R}\nu_\varphi(u)\log\left|\frac{R(z-u)}{R^2-\bar{u}z}\right|,$$

where $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, $\eta = (z_1,\dots,z_{n-1})$, $\zeta = z_n$, $z = (\eta,\zeta)$ and $|\eta| = (|z_1|^2+\dots+|z_{n-1}|^2)^{1/2}$.

Lemma 1.8. *Let h be an integrable function on $S(r)$ $(r > 0)$. Then, we have*

$$\int_{S(r)}h\sigma_n = \frac{1}{r^{2n-2}}\int_{\widetilde{B}(r)}\upsilon_{n-1}(\eta)\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}h(\eta,\zeta)\sigma_1(\zeta),$$

where $\widetilde{B}(r) := \{\eta \in \mathbb{C}^{n-1} : |\eta| < r\}$.

For a non-zero meromorphic function φ on n $B(R_0)$, there exists a zero-measure subset E of $\widetilde{B}(R_0)$ such that for each $\eta \in \widetilde{B}(R_0) \setminus E$, a meromorphic function $(\varphi|\eta)(\zeta) = \varphi(\eta, \zeta)$ is well defined on $\{\zeta \in \mathbb{C} : |\zeta| < \sqrt{R_0^2 - |\eta|^2}\}$.

Lemma 1.9. *For all $a \in P^1(\mathbb{C})$ and $0 < r < R_0$, we have*

$$\frac{1}{r^{2n-2}} \int_{\widetilde{B}(r)\setminus E} n_{\nu^a_{\varphi|\eta}}(\sqrt{r^2 - |\eta|^2})\upsilon_{n-1}(\eta) \leqq n_{\nu^a_\varphi}(r).$$

Lemma 1.10. *Let $0 < \tilde{p} < 1$ and let $0 < r < \rho < R_0$. For all $\eta \in \widetilde{B}(r) \setminus E$, we have*

$$\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} \left|\zeta \left(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\right)(\eta,\zeta)\right|^{\tilde{p}} \sigma_1(\zeta) \leqq \left(\frac{\rho}{\rho - r}\int_{|\zeta|=\sqrt{\rho^2-|\eta|^2}} |\log|\varphi(\eta,\zeta)||\sigma_1(\zeta)\right)^{\tilde{p}} + K(n_{\nu^0_\varphi}(\sqrt{\rho^2 - |\eta|^2}) + n_{\nu^\infty_\varphi}(\sqrt{\rho^2 - |\eta|^2})).$$

Proof. We can assume that $\varphi(\zeta) \neq 0, \infty$ on $\{\zeta : |\zeta| = \sqrt{\rho^2 - |\eta|^2}\}$, as each element is continuous with respect to ρ. By taking derivatives in Proposition 1.4 with application on functions $\varphi|\eta$ and $R = \tilde{\rho} := \sqrt{\rho^2 - |\eta|^2}$, we obtain

$$\left(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\right)(\eta,\zeta) = \frac{\tilde{\rho}}{\pi}\int_0^{2\pi} \frac{\log|\varphi(\eta,\tilde{\rho}e^{i\phi})|e^{i\phi}}{(\tilde{\rho}e^{i\phi} - \zeta)^2} d\phi - \sum_{|u|\leqq\tilde{\rho}} \nu_{\varphi|\eta}(u)\left\{\frac{1}{u-\zeta} - \frac{\bar{u}}{\tilde{\rho}^2 - \bar{u}\zeta}\right\}.$$

Therefore, we have

$$\left|\zeta\left(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\right)(\eta,\zeta)\right|^{\tilde{p}} \leqq \left(2\tilde{\rho}|\zeta|\int_{|u|=\tilde{\rho}} \frac{|\log|\varphi(\eta,\zeta)||}{|u-\zeta|^2}\sigma_1(u)\right)^{\tilde{p}} + \sum_{|u|\leqq\tilde{\rho}} (\nu^0_{\varphi|\eta}(u) + \nu^\infty_{\varphi|\eta}(u))\left(\left(\frac{|\zeta|}{|u-\zeta|}\right)^{\tilde{p}} + \left(\frac{|\zeta||u|}{|\tilde{\rho}^2 - \bar{u}\zeta|}\right)^{\tilde{p}}\right).$$

Taking integral and using Lemma 1.7, we can see that

$$\begin{aligned}
&\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} \left|\zeta\left(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\right)(\eta,\zeta)\right|^{\tilde\rho}\sigma_1(\zeta)\\
&\qquad \leqq \left(2\tilde\rho\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}|\zeta|\sigma_1(\zeta)\int_{|u|=\tilde\rho}\frac{|\log|\varphi(\eta,\zeta)||}{|u-\zeta|^2}\sigma_1(u)\right)^{\tilde\rho}\\
&\qquad\quad + \sum_{|u|\leqq\tilde\rho}(\nu^0_{\varphi|\eta}(u)+\nu^\infty_{\varphi|\eta}(u))\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\left(\frac{|\zeta|^{\tilde\rho}}{|u-\zeta|^{\tilde\rho}}+\frac{|\zeta|^{\tilde\rho}}{|(\tilde\rho^2/u)-\zeta|^{\tilde\rho}}\right)\sigma_1(\zeta)\\
&\qquad \leqq \left(2r\rho\int_{|u|\leqq\tilde\rho}|\log|\varphi(\eta,\zeta)||\sigma_1(u)\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\frac{1}{|u-\zeta|^2}\sigma_1(\zeta)\right)^{\tilde\rho}\\
&\qquad\quad + K\left(\sum_{|u|\leqq\tilde\rho}(\nu^0_{\varphi|\eta}(u)+\nu^\infty_{\varphi|\eta}(u))\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\frac{1}{|u-\zeta|^2}\sigma_1(\zeta) &= \int_{|\zeta|=\sqrt{r^2-|\eta|^2}}\frac{1}{|u-\zeta|^2}d^c\log|\zeta|^2\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|u-\zeta|^2}d\varphi \qquad (\text{do } d^c\log|\zeta| = \frac{1}{4\pi}d\varphi)\\
&= \frac{1}{\tilde\rho^2-(r^2-|\eta|^2)}\\
&= \frac{1}{(\rho^2-|\eta|^2)-(r^2-|\eta|^2)}\\
&= \frac{1}{\rho^2-r^2}
\end{aligned}$$

for all u such that $|u|=\tilde\rho$. Thus, we get

$$\begin{aligned}
&\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} \left|\zeta\left(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\right)(\eta,\zeta)\right|^{\tilde\rho}\sigma_1(\zeta)\\
&\qquad \leqq \left(\frac{2r\rho}{\rho^2-r^2}\int_{|u|=\tilde\rho}|\log|\varphi(\eta,u)||\sigma_1(u)\right)^{\tilde\rho} + K(n_{\nu^0_{\varphi|\eta}}(\tilde\rho)+n_{\nu^\infty_{\varphi|\eta}}(\tilde\rho)),
\end{aligned}$$

and since $0<r<\rho<R_0$ we have $\dfrac{2r\rho}{\rho^2-r^2}\leqq\dfrac{\rho}{\rho-r}$. Substituting on the

above inequality, we obtain

$$\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} \left|\zeta\left(\frac{\partial\varphi}{\partial\zeta}\Big/\varphi\right)(\eta,\zeta)\right|^{\tilde{p}}\sigma_1(\zeta) \leqq \left(\frac{\rho}{\rho-r}\int_{|\zeta|=\sqrt{\rho^2-|\eta|^2}} |\log|\varphi(\eta,\zeta)||\sigma_1(\zeta)\right)^{\tilde{p}} + K(n_{\nu_\varphi^0}(\sqrt{\rho^2-|\eta|^2}) + n_{\nu_\varphi^\infty}(\sqrt{\rho^2-|\eta|^2})).$$

This prove Lemma 1.10. □

Now we prove Theorem 1.1 for the case where $|\alpha| = 1$. We prove by induction on $|\alpha|$. First of all, we consider the case where $|\alpha| = 1$. Without loss of generation we can assume that $D^\alpha = D_n$. Let $r_0 < r < R < R_0, 0 < p < p' < 1$ and let $\tilde{p} = p/p'$, $\rho = (r+R)/2$. Since the degree of each pole of $D_n\varphi/\varphi$ is less or equal 1, $|z_n(D_n\varphi/\varphi)(z)|^{\tilde{p}}$ is integrable on $S(r)$. Using Lemmas 1.8, 1.9, 1.10 and Hölder inequality we have

$$\begin{aligned}
\int_{S(r)} |z_n(D_n\varphi/\varphi)(z)|^{\tilde{\rho}}\sigma_n(z) &\overset{1.8}{=} \frac{1}{r^{2n-2}}\int_{|\eta|\leqq r} \upsilon_{n-1}(\eta)\int_{|\zeta|=\sqrt{r^2-|\eta|^2}} |\zeta(D_n\varphi/\varphi)(\eta,\zeta)|^{\tilde{p}}\sigma_1(\zeta) \\
&\overset{1.10}{\leqq} \frac{1}{r^{2n-2}}\left(\frac{\rho}{\rho-r}\right)^{\tilde{p}}\left(\int_{|\eta|\leqq r}\upsilon_{n-1}(\eta)\right)^{1-\tilde{p}} \\
&\quad\times\left(\int_{|\eta|\leqq r}\upsilon_{n-1}(\eta)\int_{|\zeta|=\sqrt{r^2-|\eta|^2}}|\log|\varphi(\eta,\zeta)||\sigma_1(\zeta)\right)^{\tilde{p}} \\
&\quad+\frac{K}{r^{2n-2}}\int_{|\eta|\leqq r}(n_{\nu^0_{\varphi|\eta}}(\sqrt{\rho^2-|\eta|^2}) + n_{\nu^\infty_{\varphi|\eta}}(\sqrt{\rho^2-|\eta|^2}))\upsilon_{n-1}(\eta) \\
&\overset{1.9}{\leqq} \left(\frac{\rho}{\rho-r}\int_{S(\rho)}|\log|\varphi||\sigma_n\right)^{\tilde{p}} + K\left(\frac{\rho}{r}\right)^{2n-2}(n_{\nu^0_{\varphi|\eta}}(p) + n_{\nu^\infty_{\varphi|\eta}}(p)).
\end{aligned}$$

Moreover, using Lemmas 1.3, 1.6, we obtain

$$\begin{aligned}
\int_{S(r)} |z_n(D_n\varphi/\varphi)(z)|^{p}\sigma_n(z) &\leqq \left(\int_{S(r)} |z_n(D_n\varphi/\varphi)(z)|^{\tilde{\rho}}\sigma_n(z)\right)^{p'} \\
&\overset{1.10}{\leqq} \left(\frac{\rho}{\rho-r}\int_{S(\rho)}|\log|\varphi||\sigma_n\right)^{\tilde{p}p'} + K\left(\frac{\rho}{r}\right)^{2n-2}(n_{\nu^0_\varphi}(p)^{p'} + n_{\nu^\infty_\varphi}(p)^{p'}) \\
&\overset{1.6}{\leqq} \left(\frac{2R}{R-r}\int_{S(\rho)}|\log|\varphi||\sigma_n\right)^{p} + K\left(\frac{4R^{2n-1}}{R-r}(T_\varphi(R,r_0)+K)\right)^{p'} \\
&\overset{1.3}{\leqq} K\left(\frac{R^{2n-1}}{R-r}T_\varphi(r,r_0)\right)^{p'}.
\end{aligned}$$

Thus the case where $D^\alpha = D_n$ of Theorem 1.1 is proved.

To complete the proof of Theorem 1.1, we need the following lemma

Lemma 1.11. *Let φ be a non-zero meromorphic function on $B(R_0)$ and let $0 < r_0 < r < R < R_0$. Then, we have*

$$T_{D_i\varphi}(r, r_0) \leqq 3T_\varphi(r, r_0) + K \log^+ \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right),$$

with $i = 1, 2, \ldots, n$.

Proof. Since φ is meromorphic on $B(R_0)$, using Lemma 1.2 with two meromorphic functions $D_i\varphi/\varphi$ and φ we obtain:

$$T_{D_i\varphi}(r, r_0) \leqq T_{D_i\varphi/\varphi}(r, r_0) + T_\varphi(r, r_0) + K$$

Next, using Lemma 1.5 we have:

$$T_{D_i\varphi/\varphi}(r, r_0) \leqq \int_{S(r)} \log^+ |D_i\varphi/\varphi| \sigma_n + N^\infty_{D_i\varphi/\varphi}(r, r_0) + K.$$

Therefore,

$$T_{D_i\varphi}(r, r_0) \leqq \int_{S(r)} \log^+ |D_i\varphi/\varphi| \sigma_n + N^\infty_{D_i\varphi/\varphi}(r, r_0) + T_\varphi(r, r_0) + K. \quad (1)$$

On the other hand, since $\nu^\infty_{D_i\varphi/\varphi} \leqq \nu^\infty_\varphi + \nu^0_\varphi$ and using Lemma 1.4, we get

$$\begin{aligned} N^\infty_{D_i\varphi/\varphi}(r, r_0) &\leqq N^\infty_\varphi(r, r_0) + N^0_\varphi(r, r_0) \\ &\overset{1.4}{\leqq} 2T_\varphi(r, r_0) + K \quad (2). \end{aligned}$$

As we already proved Theorem 1.1 for the case where $|\alpha| = 1$, we can use Corollary 1.1 in this case. Thus, we have

$$\int_{S(r)} \log^+ |D_i\varphi/\varphi| \sigma_n \leqq K \log^+ \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right). \quad (3)$$

From (1), (2) and (3), we get:

$$T_{D_i\varphi}(r, r_0) \leqq 3T_\varphi(r, r_0) + K \log^+ \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0) \right).$$

Lemma is proved 1.11. □

Now we prove Theorem 1.1 in a general case. Suppose that Theorem 1.1 holds for all α such that $|\alpha| \leqq k$. We take α satisfying $|\alpha| = k+1$ and write $D^\alpha = D^{\alpha'} D_i$, where $1 \leqq i \leqq n$ and $|\alpha'| = k$. Then, we have $D^\alpha \varphi/\varphi = (D_i\varphi/\varphi)(D^{\alpha'}(D_i\varphi)/D_i\varphi)$, $z^\alpha = z_i z^{\alpha'}$ and $|\alpha| p = (|\alpha'|+1)p < p' < 1$. Let $p_1 := 1/(|\alpha'|+1)$ and $p_2 := |\alpha'|/(|\alpha'|+1)$. By using Hölder's inequality and the induction assumption, we get

$$\begin{aligned}
&\int_{S(r)} |z^\alpha (D^\alpha\varphi/\varphi)(z)|^p \sigma_n(z) \\
&\quad \leqq \left(\int_{S(r)} |z_i (D_i\varphi/\varphi)(z)|^{p/p_1} \sigma_n(z)\right)^{p_1} \left(\int_{S(r)} |z^{\alpha'} (D^{\alpha'}(D_i\varphi)/D_i\varphi)(z)|^{p/p_2} \sigma_n(z)\right)^{p_2} \\
&\quad \leqq K \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0)\right)^{p' p_1} \left(\frac{R^{2n-1}}{R-r} T_{D_i\varphi}(R, r_0)\right)^{p' p_2}.
\end{aligned}$$

On the other hand, as $\lim_{x\to\infty} \dfrac{\log x}{x^\alpha} = 0$ for arbitrary $\epsilon > 0$, there exists a constant K_ϵ such that

$$\log^+ \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0)\right) \leqq K_\epsilon \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0)\right)^\epsilon.$$

Using Lemma 1.11, we have:

$$\begin{aligned}
\left(\frac{R^{2n-1}}{R-r} T_{D_i\varphi}(R, r_0)\right)^{p' p_2} &\leqq \left(\frac{R^{2n-1}}{R-r}\left(3T_\varphi(R, r_0) + K_\epsilon \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0)\right)^\epsilon\right)\right)^{p' p_2} \\
&\leqq K \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0)\right)^{\epsilon_1}.
\end{aligned}$$

We choose ϵ which is small enough to satisfy $\epsilon_1 \leqq \dfrac{|\alpha'| p'}{|\alpha'|+1}$. We get $(\epsilon_1 + p' p_1) \leqq p'$. Therefore, we obtain

$$\int_{S(r)} |z^\alpha (D^\alpha\varphi/\varphi)(z)|^p \sigma_n(z) \leqq K \left(\frac{R^{2n-1}}{R-r} T_\varphi(R, r_0)\right)^{p'}.$$

This proves Theorem 1.1.

1.3 The second main theorem

Now we consider q $(\geqq N+2)$ hyperplanes

$$H_j : a_j^1 w_1 + \ldots + a_j^{N+1} w_{N+1} = 0 \ (1 \leqq j \leqq q)$$

in $P^N(\mathbb{C})$ in a general position and let

$$F_j = a_j^1 f_1 + \ldots + a_j^{N+1} f_{N+1} \ (1 \leqq j \leqq q).$$

Let $\{\alpha^1, \ldots, \alpha^{N+1}\}$ be acceptable of f. We define

$$\phi := \frac{W_{\alpha^1 \ldots \alpha^{N+1}}(f)}{F_1 F_2 \ldots F_q},$$

which is a meromorphic function on $B(R_0)$. We have

Proposition 1.5. *Let $0 < r_0 < R_0$ and let $0 < l_f t < p' < 1$. Then, there exists a constant $K > 0$ such that $r_0 < r < R < R_0$*

$$\int_{S(r)} |z^{\alpha^1 + \ldots + \alpha^{N+1}} \phi|^t \|f\|^{t(q-N-1)} \sigma_n \leqq K \left(\frac{R^{2n-1}}{R-r} T_f(R, r_0) \right)^{p'},$$

where $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ with $z = (z_1, \ldots, z_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$.

To prove Lemma 1.5, we represent here the following lemmas.

Lemma 1.12. *Let $(a_i^1, \ldots, a_i^{N+1}) \in \mathbb{C}^{N+1}$ $(i = 1, 2)$, suppose that $F_i := a_i^1 f_1 + \ldots + a_i^{N+1} f_{N+1} \not\equiv 0$ let $\varphi := \dfrac{F_1}{F_2}$. Assume that φ is a meromorphic function into $P_1(\mathbb{C})$, we have*

$$T_\varphi(r, r_0) \leqq T_f(r, r_0) + K.$$

Proof. With a reduced representation $\varphi = (g : h)$ on $B(R_0)$, $k := \dfrac{F_1}{g} = \dfrac{F_2}{h}$ is a non-zero meromorphic function. We have,

$$\|\varphi\|^2 |k|^2 = (|g|^2 + |h|^2)|k|^2 = |F_1|^2 + |F_2|^2 \leqq K \|f\|^2.$$

Therefore, we get

$$\int_{S(r)} \log \|\varphi\| \sigma_n + \int_{S(r)} \log |k| \sigma_n \leqq \int_{S(r)} \log \|f\| \sigma_n,$$

where we have used the following

$$\int_{S(r)} \log |k| \sigma_n = N_k(r, r_0) + \int_{S(r_0)} \log |k| \sigma_n \geqq \int_{S(r_0)} \log |k| \sigma_n.$$

We prove our statement. □

Lemma 1.13. *There exists a constant K such that*

$$\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(f)}{F_1\ldots F_q}\right| \|f\|^{q-N-1} \leqq K\left(\sum_{1\leqq j_1<\ldots<j_{N+1}\leqq q}\left|\frac{W_{\alpha_1\ldots\alpha_{N+1}}(F_{j_1},\ldots,F_{j_{N+1}})}{F_{j_1}\ldots F_{j_{N+1}}}\right|\right).$$

Proof. We take an arbitrary point $z \in B(R_0)$. Let $i_1,\ldots,i_q$ be a permutation of $1,2,\ldots,q$ such that

$$|F_{i_1}(z)| \leqq \ldots \leqq |F_{i_{N+1}}(z)| \leqq |F_{i_{N+2}}(z)| \leqq \ldots \leqq |F_{i_q}(z)|.$$

Since $H_1,\ldots,H_q$ are at general position, $f_1,\ldots,f_{N+1}$ are expressed linearly through $F_{i_1},\ldots,F_{i_{N+1}}$. Therefore, we can find $C_{i_1\ldots i_{N+1}}$ independently with z such that

$$|f_i(z)| \leqq C_{i_1\ldots i_{N+1}} \max_{1\leqq k\leqq N+1} |F_{i_k}(z)| \leqq C_{i_1\ldots i_{N+1}}|F_{i_l}(z)|$$

where $i = 1,\ldots,N+1$ and $l = N+2,\ldots,q$. So we have

$$\|f(z)\| = \left(\sum_{i=1}^{N+1}|f_i(z)|^2\right)^{1/2} \leqq (N+1)^{1/2}C_{i_1\ldots i_{N+1}}|F_{i_l}(z)|$$

where $l = N+2,\ldots,q$. Thus, we get

$$\|f(z)\|^{q-N-1} \leqq K_1|F_{i_{N+2}}(z)\ldots F_{i_q}(z)|,$$

where $K_1 = ((N+1)^{1/2})C^{q-N-1}_{i_1\ldots i_{N+1}}$. On the other hand, we see that

$$W_{\alpha^1\ldots\alpha^{N+1}}(f) := a_{i_1\ldots i_{N+1}}W_{\alpha^1\ldots\alpha^{N+1}}(F_{i_1},\ldots,F_{i_{N+1}})$$

with constant $a_{i_1\ldots i_{N+1}} := \det(a^j_{i_k} : 1 \leqq j,k \leqq N+1)^{-1}$. Let

$$K := \max_{1\leqq i_1<\ldots<i_{N+1}\leqq q} C_{i_1\ldots i_{N+1}}|a_{i_1\ldots i_{N+1}}|,$$

we have

$$\begin{aligned}\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(f)}{F_1\ldots F_q}(z)\right| \|f(z)\|^{q-N-1} &\leqq C_{i_1\ldots i_{N+1}}|a_{i_1\ldots i_{N+1}}|\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(F_{i_1},\ldots,F_{i_{N+1}})F_{i_{N+2}}\ldots F_{i_q}}{F_1F_2\ldots F_q}(z)\right| \\ &\leqq K\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(F_{i_1},\ldots,F_{i_{N+1}})}{F_{i_1}\ldots F_{i_{N+1}}}(z)\right| \\ &\leqq K\left(\sum_{1\leqq i_1<\ldots<i_{N+1}\leqq q}\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(F_{i_1},\ldots,F_{i_{N+1}})}{F_{i_1}\ldots F_{i_{N+1}}}(z)\right|\right).\end{aligned}$$

This prove our statement. □

Now we prove Lemma 1.5

Proof. We denote $l(k) = \mathrm{rank}_{\mathcal{M}_p}\mathcal{F}_p^k$. Then, by Remark 1.1 there is an integer number k_0 such that $l(k_0) = N + 1$,

$$\begin{aligned} l_f &= \sum_{k \geqq 0} k r_f(k) \\ &= \sum_{k \geqq 0}^{k_0} k r_f(k) \\ &= (l(k_0) - l(k_0 - 1)) + (l(k_0) - l(k_0 - 2)) + \ldots + (l(k_0) - l(0)) \\ &= \gamma_{k_0 - 1} + \ldots + \gamma_0 \text{ with } \gamma_j = (l(k_0) - l(k_0 - k_j)). \end{aligned}$$

Without loss of generation, we take $\gamma_j = (\gamma_j, 0, \ldots, 0), \forall j = 0, 1, \ldots, k_0 - 1$. By Lemma 1.13, we have

$$\left| \frac{W_{\alpha^1 \ldots \alpha^{N+1}}(f)}{F_1 \ldots F_q} \right| \|f\|^{q-N-1} \leqq K \left(\sum_{1 \leqq j_1 < \ldots < j_{N+1} \leqq q} \left| \frac{W_{\alpha^1 \ldots \alpha^{N+1}}(F_{j_1}, \ldots, F_{j_{N+1}})}{F_{j_1} \ldots F_{j_{N+1}}} \right| \right),$$

where K is constant. On the other hand, let $\alpha = \alpha^1 + \ldots + \alpha^{N+1}$ then

$$\begin{aligned} I := \int_{S(r)} \left| z^{\alpha} \frac{W_{\alpha^1 \ldots \alpha^{N+1}}(F_{j_1}, \ldots, F_{j_{N+1}})}{F_{j_1} \ldots F_{j_{N+1}}} \right|^t \sigma_n &\leqq \int_{S(r)} \left| z^{\alpha - l_f} \right|^t \left| z^{l_f} \frac{W_{\alpha^1 \ldots \alpha^{N+1}}(F_{j_1}, \ldots, F_{j_{N+1}})}{F_{j_1} \ldots F_{j_{N+1}}} \right|^t \sigma_n \\ &\leqq K \int_{S(r)} \left| z^{l_f} \frac{W_{\alpha^1 \ldots \alpha^{N+1}}(F_{j_1}, \ldots, F_{j_{N+1}})}{F_{j_1} \ldots F_{j_{N+1}}} \right|^t \sigma_n. \end{aligned}$$

So we only consider the following term:

$$\int_{S(r)} \left| z^{\gamma_0 + \ldots + \gamma_{k_0 - 1}} \frac{W_{\alpha^1 \ldots \alpha^{N+1}}(F_{j_1}, \ldots, F_{j_{N+1}})}{F_{j_1} \ldots F_{j_{N+1}}} \right|^t \sigma_n,$$

with $1 \leqq j_1 < \ldots < j_{N+1} \leqq q$. The expression under the integral sign can be approximated by multiples of positive constant of sums of functions which have the following form

$$\psi_{i_0 \ldots i_{k_0 - 1}} := \left| z^{\gamma_0 + \ldots + \gamma_{k_0 - 1}} \frac{D^{\gamma_0} \varphi_{i_0}}{\varphi_{i_0}} \cdots \frac{D^{\gamma_{k_0 - 1}} \varphi_{i_{k_0 - 1}}}{\varphi_{i_{k_0 - 1}}} \right|^t,$$

where $\varphi_i = \dfrac{F_i}{F_1}$ $(1 \leqq i \leqq q)$ and $1 \leqq i_0, \ldots, i_{k_0 - 1} \leqq q$.

Denote $p_j = \dfrac{\gamma_j}{l_f}$ with $0 \leqq j \leqq k_0 - 1$. By the Hölder's inequality, we have:

$$\int_{S(r)} \psi_{i_0 \dots i_{k_0-1}} \sigma_n \leqq \prod_{j=0}^{k_0-1} \left(\int_{S(r)} \left| z^{\gamma_j} \frac{D^{\gamma_j} \varphi_{i_j}}{\varphi_{i_j}} \right|^{\frac{t}{p_j}} \sigma_n \right)^{p_j}.$$

We see that $(\frac{t}{p_j})|\gamma_j| = (\gamma_0 + \ldots + \gamma_{k_0-1})t = l_f t < p' < 1$ where $0 \leqq j \leqq k_0 - 1$, using the Lemma on logarithmic derivative we obtain

$$\int_{S(r)} \psi_{i_0 \dots i_{k_0-1}} \sigma_n \leqq K \prod_{j=0}^{k_0-1} \left(\frac{R^{2n-1}}{R-r} T_{\varphi_{i_j}}(R, r_0) \right)^{p' p_j}.$$

On the other hand, using Lemma 1.12, we get

$$T_{\varphi_i}(r, r_0) \leqq T_f(r, r_0) + K,$$

for all $i = 1, 2, \ldots, q$. Thus, we have

$$I \leqq K \left(\frac{R^{2n-1}}{R-r} T_f(R, r_0) \right)^{p'}.$$

□

For real-valued functions $f(r)$ and $g(r)$ on $[r_0, R_0)$ we denote $\| f(r) \leqq g(r)$, i.e. $f(r) \leqq g(r)$ on $[r_0, R_0)$ except for a set E such that $\int_E dr < \infty$ if $R_0 = \infty$ and $\int_E (R_0 - r)^1 dr < \infty$ if $R_0 < \infty$. By Lemma 1.5 we have the Second main theorem which is stated as follows.

Theorem 1.2 (The second main theorem)**.** *Let $f : B(R_0) \to P^N(\mathbb{C})$ be a non-degenerate meromorphic function and let $H_1, \ldots, H_q$ be hyperplanes at general position. Then, we have*

$$(q - N - 1) T_f(r, r_0) \leqq N_\phi^\infty(r, r_0) + S_f(r),$$

where there exists a constant K satisfying

$$\| S_f(r) \leqq l_f \log \frac{1}{R_0 - r} + K \log^+ T_f(r, r_0) \text{ if } R_0 < \infty,$$

$$\| S_f(r) \leqq K(\log^+ T_f(r, r_0) + \log r) \text{ if } R_0 = \infty.$$

Proof. By Lemma 1.5 and the convexity of a logarith function, we have

$$\begin{aligned} t\int_{S(r)} \log\left|z^{\alpha^1+\ldots+\alpha^{N+1}}\right|\sigma_n + t\int_{S(r)} \log\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(f)}{F_1\ldots F_q}\right|\sigma_n & \\ + t(q-N-1)\int_{S(r)} \log\|f\|\sigma_n & \\ \leqq \log\int_{S(r)} \left|z^{\alpha^1+\ldots+\alpha^{N+1}}\frac{W_{\alpha^1\ldots\alpha^{N+1}}(f)}{F_1\ldots F_q}\right|^t \|f\|^{t(q-N-1)}\sigma_n & \\ \leqq \log K + p'\log\left(\frac{R^{2n-1}}{R-r}T_f(R,r_0)\right) & \\ = \log K + p'\left(R^{2n-2}\log\frac{R}{R-r} + \log T_f(R,r_0)\right) & \\ \leqq K\left(\log^+\frac{R}{R-r} + \log^+ T_f(R,r_0)\right). & \end{aligned}$$

Using Jensen's formula, we get

$$-N_\phi^\infty(r,r_0) \leqq \int_{S(r)} \log\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(f)}{F_1\ldots F_q}\right|\sigma_n$$

$$\Rightarrow -\int_{S(r)} \log\left|\frac{W_{\alpha^1\ldots\alpha^{N+1}}(f)}{F_1\ldots F_q}\right|\sigma_n \leqq N_\phi^\infty(r,r_0).$$

On the other hand, we have

$$T_f(r,r_0) = \int_{S(r)} \log\|f\|\sigma_n - \int_{S(r_0)} \log\|f\|\sigma_n.$$

Therefore,

$$(q-N-1)T_f(r,r_0) \leqq N_\phi^\infty(r,r_0) + K(\log^+\frac{R}{R-r} + \log^+ T_f(R,r_0)).$$

Let

$$S_f(r) = K(\log^+\frac{R}{R-r} + \log^+ T_f(R,r_0)). \quad (*)$$

Since $T_f(r,r_0)$ is a continuous ,increasing function and we can assume that $T_f(r,r_0) \geqq 1$. Using Lemma 2.4 in Hayman, we obtain:

$$T_f\left(r + \frac{R_0 - r}{eT_f(r,r_0)}, r_0\right) \leqq 2T_f(r,r_0)$$

except for a set E such that $\int\limits_E \frac{1}{R_0 - r} dr < \infty$ and

$$T_f\left(r + \frac{1}{T_f(r, r_0)}, r_0\right) < 2T_f(r, r_0)$$

except for the set E' such that $\int_E dr < \infty$ in the case where $R_0 = \infty$.

Substituting in (*) $R = r + \dfrac{1}{T_f(r, r_0)}$, if $R_0 = \infty$ we have:

$$\begin{aligned}
\|S_f(r) &= K\left(\log r + \log(rT_f(r, r_0) + 1) + \log^+ T_f(r + \frac{1}{T_f(r, r_0)}, r_0)\right) \\
&\leqq K\left(\log r + \log^+ T_f(r, r_0)\right).
\end{aligned}$$

If $R_0 < \infty$, thay $R = r + \dfrac{R_0 - r}{eT_f(r, r_0)}$, we obtain:

$$\begin{aligned}
\|S_f(r) &= K\left(\log^+ \frac{reT_f(r, r_0) + R_0 - r}{R_0 - r} + \log^+ T_f\left(r + \frac{R_0 - r}{eT_f(r, r_0)}, r_0\right)\right) \\
&\leqq K\left(\log^+ reT_f(r, r_0) + \log 2 + \log\frac{1}{R_0 - r} + \log^+ T_f(r, r_0)\right) \\
&\leqq l_f \log\frac{1}{R_0 - r} + K\log^+ T_f(r, r_0).
\end{aligned}$$

□

Notation 1.1. In Theorem 1.2, if $R_0 = \infty$ and $\lim_{r\to\infty} \frac{T_f(r, r_0)}{\log r} < \infty$, or equivalently f is rational, then we can choose $S_f(r)$ such that it is bounded.

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